Optimization and Control of Networks

Path Algebra

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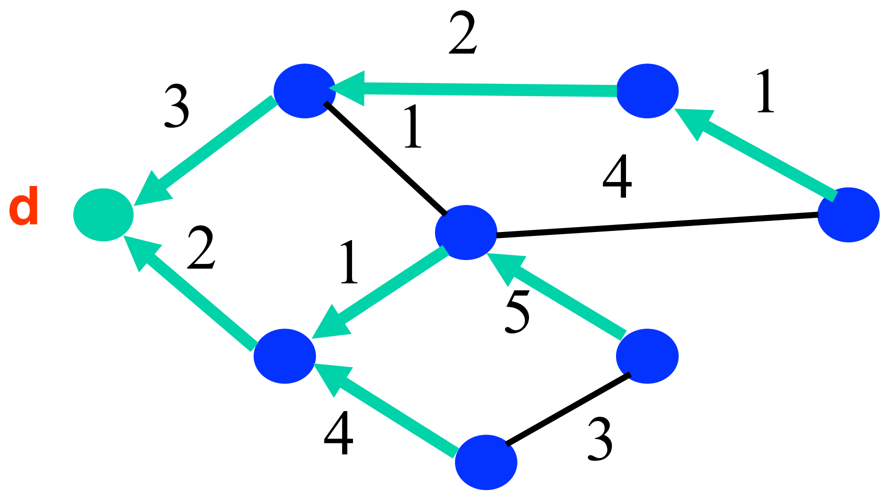
Algebra framework for network routing

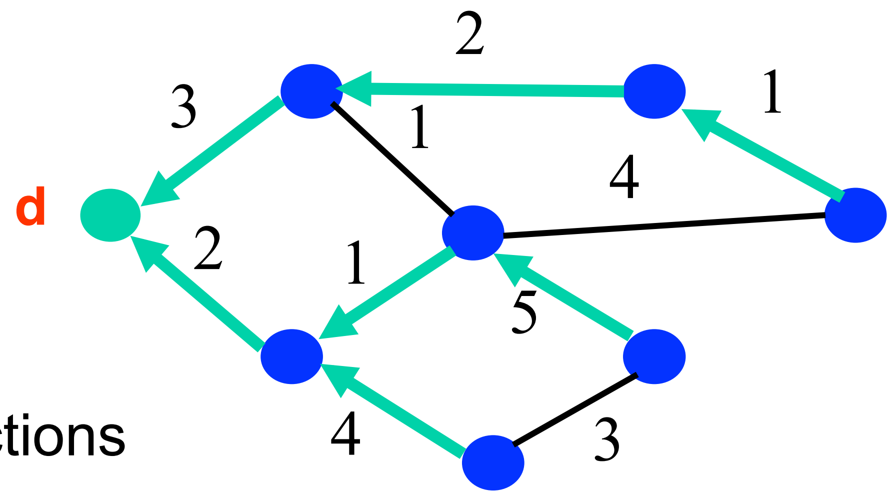
- Goal
  - Convergence properties of dynamic routing protocols
  - Characteristics of the paths the protocol converges to
- Approach
  - Formulation in an algebraic framework
  - Different protocols can be seen as different instantiations of this algebra

Shortest path routing as a path vector protocol

- Given a weighted graph G=(V,E,w), find the shortest paths to a destination.



Shortest path routing as a path vector protocol



- Abstractions
  - Each link is assigned a weight (**label**)
  - Each path has an aggregate weight (**signature**)
  - Path extension: path weight is the summation of weights of all its links (**composition operation**)
  - Preference/selection rule: choose the minimum weight path (**ordering/preference**)

Shortest path algebra

Set of labels: L ←	R^+
Set of signatures: Σ ←	$R^+ \cup \{+\infty\}$
Special signature: ϕ ←	$+\infty$
Totally ordered set of weights: $(W, \preceq)$ ←	$(R^+ \cup \{+\infty\}, \leq)$
Composition: $\oplus : L \times \Sigma \rightarrow \Sigma$ ←	$+$
Weighing function: $f : \Sigma \rightarrow W$ ←	Identity

Path vector algebra

- Defined as a seven-tuple $(w, \preceq, L, \Sigma, \phi, \oplus, f)$
- Labels model link data-structure, $\oplus$ amounts to applying edge policy to the path data-structure when the path is extended
- Signatures model the path data-structure, contain enough information to determine a path's weight using f
- Weight ordering defines preference relation, heavier paths are less preferred

Path Vector protocol

- Instantiating an algebra produces a protocol
- A node v knows a path P to d when having a signature for P (either $s(d)$ for trivial path, or $s(P)$ for a path extending a neighbor's path)
- The best path to d is a path with lowest weight
- To advertise a path P to node u, $s(P)$ is sent along signaling edge (v,u) with some associated label l, and $s(uP) = l \oplus s(P)$ is the signature of the imported, extended path at u

What we care

- Convergence is the most important goal
- Characteristics of the paths the protocol converges to (optimal? In what sense?)
- SP routing converges and is optimal. Let’s first check what properties SP algebra has, and then abstract them to general algebra and see whether they are necessary and/or sufficient for the convergence and optimality.

Maximality and Absorption

- Usable path has finite weight
  - Maximality $\forall_{\alpha \in \Sigma - \{\phi\}} f(\alpha) \prec f(\phi)$
- Cannot extend an un-usable path to a usable one
  - Absorption

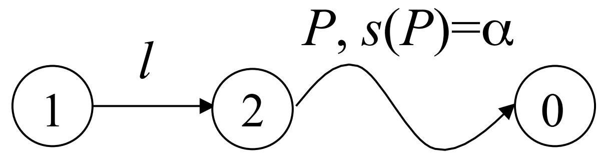
$$\forall_{l \in L} l \oplus \phi = \phi$$

Monotonicity

- The path weight is non-decreasing along the path

- For every label l and signature α

$$f(\alpha) \preceq f(l \oplus \alpha)$$



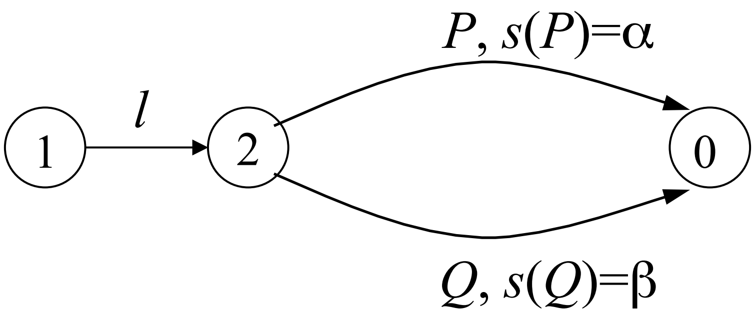
$1\,2 \circ P$ does not weigh less than P

Isotonicity

- Path ordering is kept when extended onto common link

  - For every label l and signatures α and β

$$f(\alpha) \preceq f(\beta) \Rightarrow f(l \oplus \alpha) \preceq f(l \oplus \beta)$$

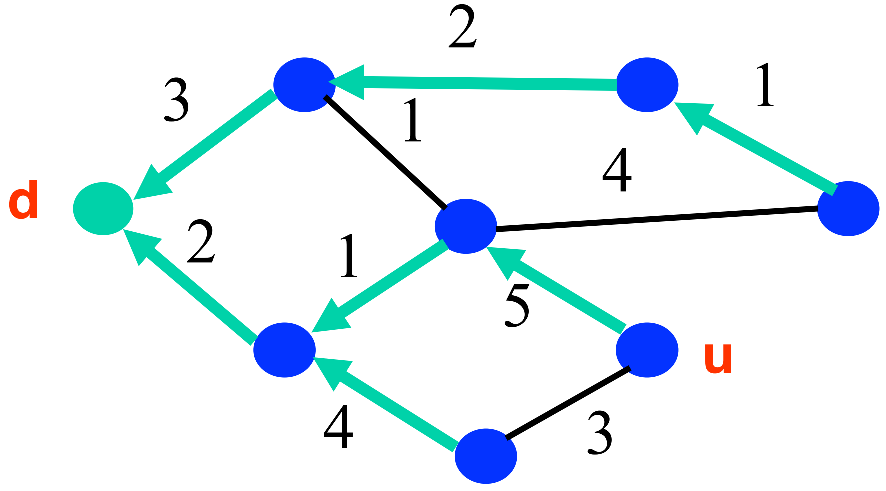


If P does not weigh more than Q, then

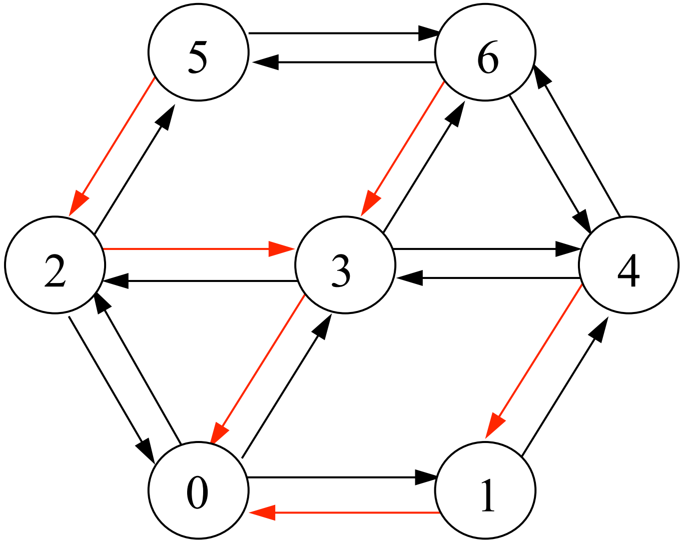
$1\,2 \circ P$ does not weigh more $1\,2 \circ Q$

Some Definitions

- An optimal path from u to d is a usable path with minimum weight among all usable paths from u to d
- An optimal path in-tree rooted at node d is an in-tree routed at d satisfying, for every node u that belongs to the in-tree, the only path in the in-tree is an optimal path



- The local-optimal path from u to d exists only with respect to a set of paths, which are advertised to u by its neighbors.
- Given in-tree T_d rooted at d, define $V_u(T_d)$ as the set of in-tree paths which has an out-neighbor of u for origin
- In-tree T_d is local-optimal-paths in-tree satisfying, for each of its node u, the only path in the in-tree is a local-optimal path with respect to $V_u(T_d)$



Freeness

$$\forall_{cycle} u_n \cdots u_1 u_0 \;\; \forall w \in W - \{f(\phi)\} \; \exists 0 < i \le n \;\; \forall \alpha \in \Sigma$$
$$f(\alpha) = w \Rightarrow f\left(l(u_i u_{i-1}) \oplus \alpha\right) \neq w$$

given a cycle and a set of paths with origins at the nodes of the cycle, all with the same weight, at least one of these paths will see its weight change as it extends into the cycle.

for example, in shortest-path routing, the network is free, since the algebra is strictly monotone

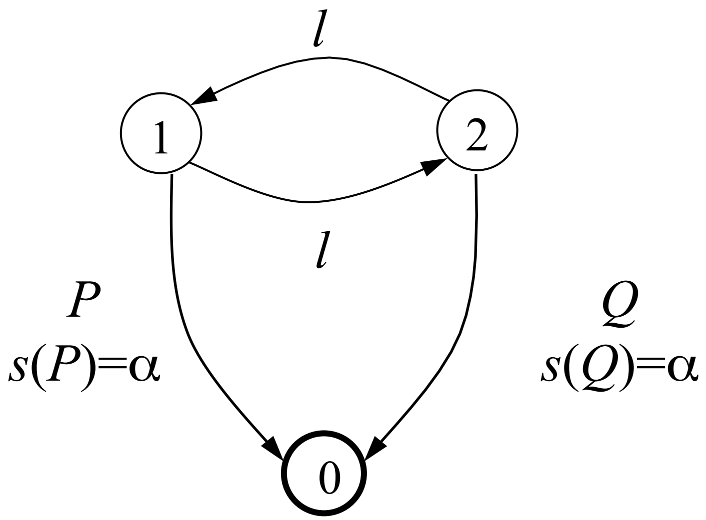
Convergence

Network is free

Convergence to local-optimal in-tree

If the algebra is monotone, then the path vector protocol can be made to converge to local-optimal in-tree.

Failure of monotonicity



$$f(\alpha) \succ f(l \oplus \alpha)$$

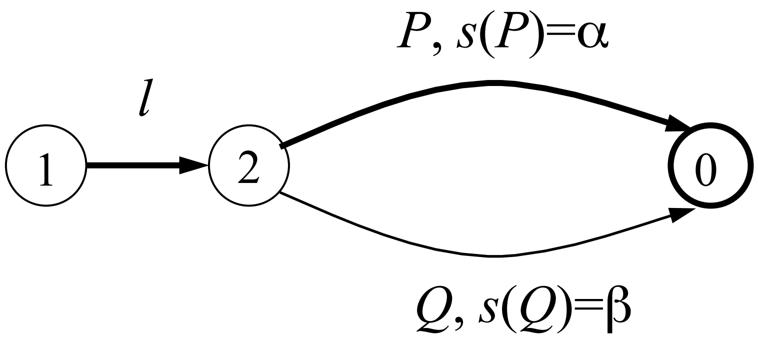
Isotonicity and optimality

Algebra is isotone ⟺ Local-optimal in-trees = Optimal in-trees

If the signature/weight depends on some global property of the path, we may say something about it by this theorem. For example, SP routing is optimal.

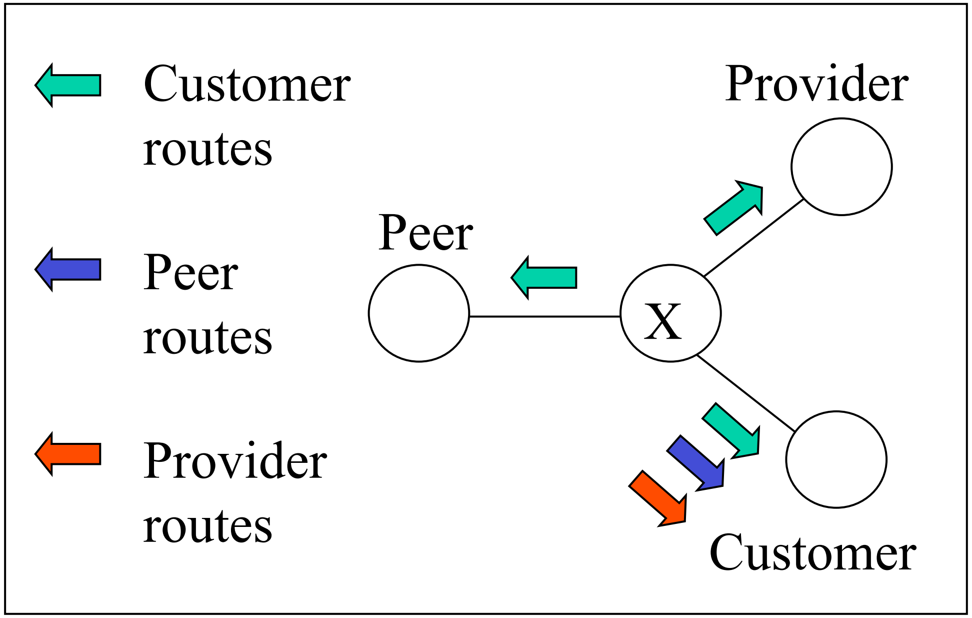
Failure of isotonicity

$$f(\alpha) \preceq f(\beta) \text{ but } f(l \oplus \alpha) \succ f(l \oplus \beta)$$



A local-optimal-path in-tree that is not optimal

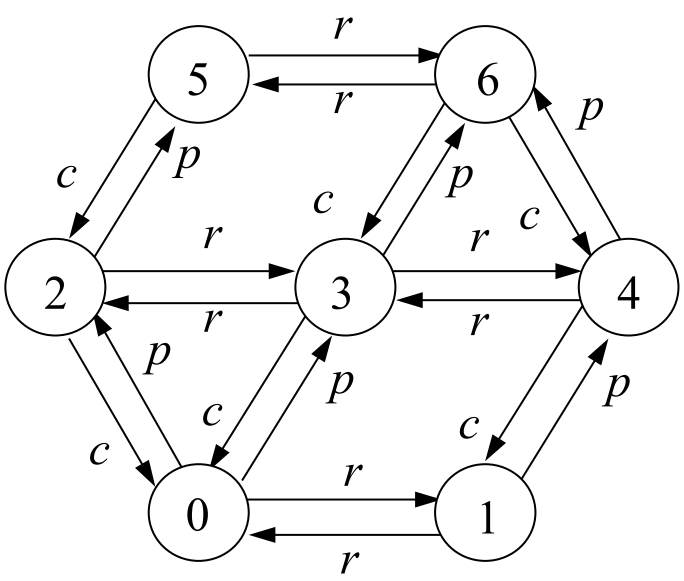
Gao-Rexford: import/export rules



- Prefer customer routes to peer routes, and then to provider routes
- Export only customer routes to peers/providers, and export all kinds of routes to customers

Algebra: labels

- Each link carries a label
  - Customer link, *c*
  - Provider link, *p*
  - Peer link, *r*



3 is a provider of 0; 0 is a customer of 3

3 and 4 are peers

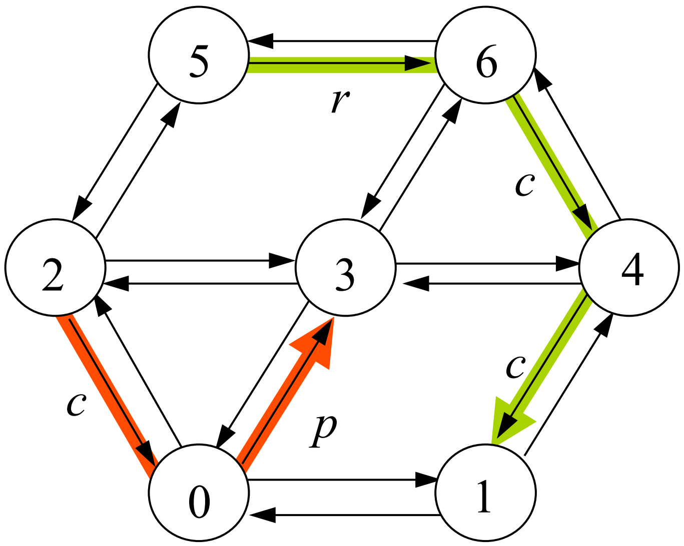
Algebra: signatures

- Each path has a signature
  - Trivial path, ε
  - Unusable path, ϕ
  - Customer path, *c*
  - Provider path, *p*
  - Peer path, *r*

Algebra: composition

Signature (columns) / Label (rows)

$\oplus$	ε	c	r	p	ϕ
c	c	c	ϕ	ϕ	ϕ
r	r	r	ϕ	ϕ	ϕ
p	p	p	p	p	ϕ



- A node does not export to a provider a route learned from another provider
- A node exports to its peers routes learned from its customers

Algebra: weights

$0 \quad 1 \quad 2 \quad 3 \quad +\infty$ weights

f

$\oplus$	ε	c	r	p	ϕ
c	c	c	ϕ	ϕ	ϕ
r	r	r	ϕ	ϕ	ϕ
p	p	p	p	p	ϕ

- Prefer customer routes to peer routes and then to provider routes

Algebra: properties

f	0	1	2	3	$+\infty$
$\oplus$	ε	c	r	p	ϕ
c	c	c	ϕ	ϕ	ϕ
r	r	r	ϕ	ϕ	ϕ
p	p	p	p	p	ϕ

- The algebra is
  - Monotone
  - Isotone

Network: freeness

- L0 is empty, since $0 = f(\varepsilon) \prec f(l \oplus \varepsilon)$
- L1={c}, since $1 = f(c) = f(c \oplus c)$
- L2 is empty, since $2 = f(r) \neq f(l \oplus r)$
- L3={p}, since $3 = f(p) = f(p \oplus p)$

- Non-free cycles:
  - Only customer links
  - Only provider links

Gao-Rexford: convergence

- The protocol converges, independent of whatever preference given to the paths with the same weight, if the network is free.
- Means that the ordering within one class (customer, provider) is not important for convergence. Any preference given to them will result in convergence
- Since the algebra is also isotone, the path is optimal. But this says nothing about global property of the path, since the weight is only decided by the first link of the paths.

References

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