Optimization and Control of Networks

Potential Games and the Inefficiency of Equilibrium

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Outline

- Potential games
  - Review on strategic games
  - Potential games (atomic and nonatomic)
- Inefficiency of equilibrium
  - The price of anarchy and selfish routing
  - Resource allocation
  - Network design games and the price of stability

Strategic game

- Def: a game in strategic form is a triple

$$G = \{N, S_{i \in N}, u_{i \in N}\}$$

  - N is the set of players (agents)
  - S_i is the player i strategy space
  - $u_i : S \to R$ is the player i payoff function
- Notations
  - $S = S_1 \times S_2 \times \cdots \times S_N$: the set of all profiles of player strategies
  - $s = (s_1, s_2, \cdots, s_N)$: profile of strategies
  - $s_{-i} = (s_1, s_2, \cdots, s_{i-1}, s_{i+1}, \cdots, s_N)$: the profile of strategies other than player i

Potential games (atomic)

- Definition: A function $\Phi : S \rightarrow R$ is a potential function for game G if for $\forall i, \forall s_{-i} \in S_{-i}, \forall s_i, \bar{s}_i \in S_i$,

$$u_i(s_i, s_{-i}) - u_i(\bar{s}_i, s_{-i}) = \Phi(s_i, s_{-i}) - \Phi(\bar{s}_i, s_{-i}).$$

  When Φ exists, the game is called a potential game.

- Definition: A function $\Phi : S \rightarrow R$ is an ordinal potential function for game G if for $\forall i, \forall s_{-i} \in S_{-i}, \forall s_i, \bar{s}_i \in S_i$,

$$u_i(s_i, s_{-i}) - u_i(\bar{s}_i, s_{-i}) > 0 \Leftrightarrow \Phi(s_i, s_{-i}) - \Phi(\bar{s}_i, s_{-i}) > 0.$$

  When Φ exists, the game is called an ordinal potential game.

- Example

Game

1,1	3,0
0,3	2,2

Potential

2	1
1	0

Equilibrium

- s^* is a pure strategy Nash equilibrium for ordinal potential game G, iff

$$\Phi(s_i^*, s_{-i}^*) \geq \Phi(s_i, s_{-i}^*), \forall i, \forall s_i \in S_i.$$

- Proof: If Φ is a potential function

$$u_i(s_i^*, s_{-i}^*) - u_i(s_i, s_{-i}^*) = 0 \Leftrightarrow \Phi(s_i^*, s_{-i}^*) - \Phi(s_i, s_{-i}^*) = 0.$$

s^* is a pure strategy Nash equilibrium, iff

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$$

$$\Leftrightarrow \Phi(s_i^*, s_{-i}^*) \geq \Phi(s_i, s_{-i}^*).$$

- If Φ has a maximum at s^*, then s^* is a pure strategy Nash equilibrium of the ordinal game.
- Every finite ordinal potential game has a pure strategy Nash equilibrium.
- Continuous ordinal potential game has a pure strategy Nash equilibrium if the strategy space is compact and potential is continuous.

Congestion games

- Definition: A congestion model $\{N, M, S_{i\in N}, c_{j\in M}\}$ is defined as follows
  - N is the set of players
  - M is the set of facilities or resources
  - S_i is the sets of the resources that player i can use
  - $c_j(k_j)$ is the cost to users who use the resource j when k_j users are using it
- Definition: A congestion game associated with a congestion model is a game $\{N, S_{i\in N}, c_{i\in N}\}$ with cost

$$c_i(s) = \sum_{j\in s_i} c_j(k_j)$$

- Every congestion game is a potential game, with potential

$$\Phi(s) = \sum_{j \in \cup s_i} \sum_{k=1}^{k_j} c_j(k).$$

- Congestion games have many applications
  - Network design

Potential games (nonatomic)

- Nonatomic game: the user number is infinite
  - N classes of "infinitesimal" players
  - r_i the "mass" of class i players
  - $f(i,s_i)$ the fraction of class i players that choose strategy s_i
  - $u_i(s_i;f)$ the payoff for a player of class i with s_i
- Definition: f^* is an equilibrium if for all $\forall i, \forall s_i, \bar{s}_i \in S_i$,

$$f^*(i,s_i) > 0 \Rightarrow u_i(s_i;f^*) \geq u_i(\bar{s}_i;f^*).$$

- Definition: A nonatomic game is a potential game if there exists potential function $\Phi(f)$ such that

$$u_i(s_i;f) = \frac{\partial \Phi(f)}{\partial f(i,s_i)}.$$

Example: selfish routing

- Consider a multicommodity flow network (V,E)
  - N source-destination pairs (commodities)
  - Each commodity i has a total rate r_i, and can use a set of paths P_i
  - The aggregate traffic among link e

$$f_e = \sum_{i,s_i:e\in s_i} f(i,s_i)$$

  - $c_e(f_e)$ link e cost, a nonnegative, continuous, non-decreasing function of traffic f_e
  - The cost $c_i(s_i;f) = \sum_{e\in s_i} c_e(f_e)$

- Wardrop equilibrium: the costs of all the paths actually used are equal, and less than those which would be experienced by a single user on any unused path.
- $\{V, E; r, c\}$ is a potential game with potential

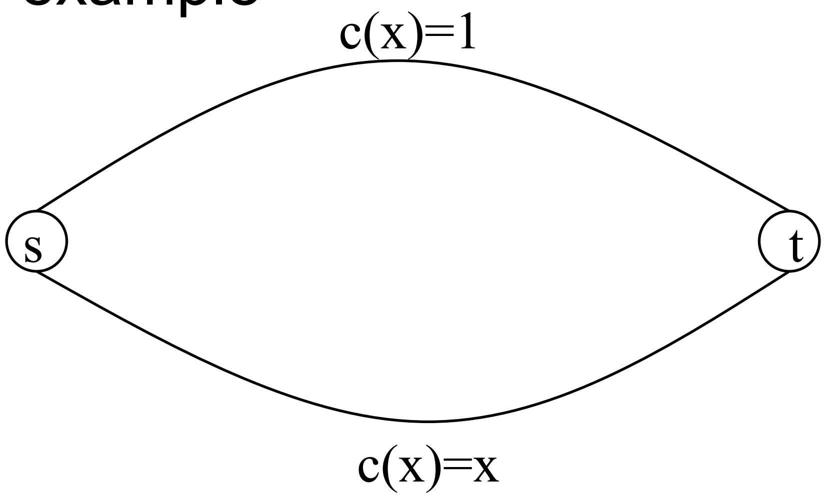
$$\Phi(f) = \sum_e \int_0^{f_e} c_e(x)\,dx.$$

Inefficiency of equilibria

- Equilibria of strategic games are typically inefficient
- Example: Prisoner's Dilemma

	D	C
D	3,3	0,4
C	4,0	1,1

- Pigou's example



One commodity with rate 1

- A unique Wardrop equilibrium, with all traffic routed on the lower edge
- A better flow: route half of the traffic on each of the two edges

Questions we care

- The existence of equilibrium
- Quantify the inefficiency of equilibrium
- The convergence of equilibrium

Quantify the inefficiency

- Price of anarchy: quantify inefficiency with respect to some objective function

$$\text{price of anarchy} = \frac{\text{obj fn value of the worst equilibrium}}{\text{optimal obj fn value}}$$

- <=1 for maximization; >=1 for minimization
- Interested in situations in which we can bound the price of anarchy

Selfish Routing

- At Wardrop equilibrium

$$f(i,s_i)(c_i(s_i;f)-c_i(f))=0$$

$$c_i(s_i;f)-c_i(f)\geq 0$$

$$\sum_{s_i} f(i,s_i)=r_i$$

$$c_i(f)=\min_{s_i} c_i(s_i,f)$$

- The above is the KKT optimality condition for

$$\min_f \quad \Phi(f)=\sum_e \int_0^{f_e} c_e(x)dx$$

$$\text{s.t.} \quad \sum_{s_i} f(i,s_i)=r_i$$

$$\sum_{i,s_i:e\in s_i} f(i,s_i)=f_e$$

- A flow f for $\{V, E; r, c\}$ is a Wardrop equilibrium if and only if it is a global minimum of the potential function

$$\Phi(f) = \sum_e \int_0^{f_e} c_e(x)\,dx.$$

- Define the objective function, i.e., the cost of flow as

$$C(f) = \sum_{i, s_i} f(i, s_i) c_i(s_i, f) = \sum_e f_e c_e(f_e)$$

- Definition: An optimal flow f^* for $\{V, E; r, c\}$ is the flow that minimizes $C(f)$.

The price of anarchy

- The price of anarchy is

$$\rho = \frac{C(f)}{C(f^*)}.$$

- Pigou's example $\rho = 4/3$
- Suppose that $x \cdot c_e(x) \le \gamma \cdot \int_0^x c_e(y)dy$, then $\rho \le \gamma$.
- Pigou's example with degree-d polynomial cost
  - $x \cdot c_e(x) \le (d+1) \cdot \int_0^x c_e(y)dy$
  - $\rho \le d + 1$

Resource allocation

- Consider a simple network: the sources (users) share a link and the network (link) manager wants to allocate link rate such that

$$\textit{System:} \qquad \max_{x} \sum_{s} U_s(x_s)$$
$$s.t. \quad \sum_{s} x_s \le c$$

- Utility functions are not known to the link manager

Market-clearing mechanism

- Each user s submits a bid (or willingness to pay) w_s
- the manager seeks to allocate the entire link capacity, and sets a price p such that

$$\sum_s \frac{w_s}{p} = c$$

  - As if the user has a demand function

$$D(p, w_s) = w_s / p$$

  - The link manager chooses a price to clear the market

$$\sum_s D(p, w_s) = c$$

Price taking users and competitive equilibrium

- The user is a price taker: does not anticipate the effect of his payment on the price
- It is rational for the user to maximize the following payoff (Kelly ’98)

$$u_s(p, w_s) = U_s\left(\frac{w_s}{p}\right) - w_s$$

- A pair (p, w) is a competitive equilibrium if

$$u_s(p, w_s) \geq u_s(p, \overline{w}_s) \text{ for any } \overline{w}_s \geq 0$$
$$p = \left(\sum_s w_s\right)/c$$

- Theorem (Kelly ’98): there exist a unique competitive equilibrium (p, w) such that $x = w/p$ solves the problem system.
- Proof: consider the Lagrangian

$$D(p,x) = \sum_s U(x_s) - p(\sum_s x_s - c)$$

  - At primal-dual optimal

$$U_s'(x_s) = p,\ \text{if}\ x_s > 0$$

$$U_s'(x_s) \le p,\ \text{if}\ x_s = 0$$

$$p \ge 0$$

$$p(\sum_s x_s - c) = 0$$

- Since $c > 0$, at least one x_s is positive. So, $p > 0$.
- Thus, $\sum_s x_s = c$.
- Let $w = px$, then (p, w) is a competitive equilibrium and $x = w / p$ solves the problem *System*.
- In this case, the uniqueness of x follows from the uniqueness of p.

Price anticipating users and Nash equilibrium

- Price anticipating users realizes that the price is set according to $p = (\sum_s w_s)/c$, and will adjust their bids accordingly.
- This makes the model a game, where user payoff is (Johari ' 04)

$$u_s(w_s, w_{-w}) = \begin{cases} U_s(\frac{w_s}{\sum_s w_s} c) - w_s, \text{ if } w_s > 0 \\ U_s(0), \qquad\qquad \text{if } w_s = 0 \end{cases}$$

- Consider Nash equilibrium w such that

$$u_s(w_s, w_{-s}) \geq u_s(\overline{w}_s, w_{-s}), \text{ for all } \overline{w} \geq 0, \text{ for all } s$$

- Theorem (Hajek, et al): there exists a unique Nash equilibrium $w \geq 0$. Moreover, the rates $x_s = \frac{w_s}{\sum_s w_s} c$ are unique solution of the following problem

$$Game: \quad \max_x \sum_s \hat{U}_s(x_s)$$

$$s.t. \quad \sum_s x_s \leq c$$

where

$$\hat{U}_s(x_s) = (1 - \frac{x_s}{c}) U_s(x_s) + \frac{x_s}{c} (\frac{1}{x_s} \int_0^{x_s} U_s(z) dz).$$

- Proof:
  - If w is a Nash equilibrium, at least two players have nonzero bids.
  - Then $u_s(w_s, w_{-w})$ is strictly concave and continuously differentiable in w_s .
  - Then, at equilibrium

$$U_s^{'}\left(\frac{w_s}{\sum_t w_t}c\right)\left(1-\frac{w_s}{\sum_t w_t}\right)=\frac{\sum_t w_t}{c},\ \text{ if }\ w_s>0$$

$$U_s^{'}(0)\le\frac{\sum_t w_t}{c},\ \text{ if }\ w_s=0$$

  - The above condition is also sufficient.

- The problem *Game* has a unique optimal x. Moreover, there exist a p such that

$$U_s'(x_s)(1-\frac{x_s}{c}) = p, \text{ if } x_s > 0$$

$$U_s'(x_s) \le p, \text{ if } x_s = 0$$

$$p \ge 0$$

$$p(\sum_s x_s - c) = 0$$

- Let $x = w/p$ and $p = \sum_t w_t / c$. Then (x, p) satisfies the above optimality condition.

The price of anarchy

- Assume $U_s(0) \geq 0$, we have

$$\frac{1}{x_s}\int_0^{x_s} U_s(z)dz \leq U_s(x_s)$$

- Then $\hat{U}_s(x_s) \leq U_s(x_s)$
- Since $U_s(z) \geq \frac{z}{x_s}U_s(x_s) + (1-\frac{z}{x_s})U_s(0),\ 0 \leq z \leq x_s$, we have

$$\int_0^{x_s} U_s(z)dz \geq \frac{x_s}{2}U_s(x_s)$$

- Then, $\hat{U}_s(x_s) \geq \frac{1}{2}U_s(x_s)$

- Let x^* and x are the optima of problems *System* and *Game*, we have

$$\frac{1}{2}\sum_s U_s(x_s^*) \le \sum_s \hat{U}_s(x_s^*) \le \sum_s \hat{U}_s(x_s) \le \sum_s U_s(x_s)$$

- The price of anarchy $\rho \ge 1/2$

Tight bound

- Define the JT bound β by

$$\beta = \inf_U \inf_c \inf_{0 \le x, x^* \le c} \frac{U(x) + \hat{U}'(x)(x^* - x)}{U(x^*)}$$

- For any $\varepsilon > 0$, there is a resource allocation game with the price of anarchy at most $\beta + \varepsilon$.
  - Proof: first note that we can assume $x < x^*$ & $c = x^*$.
  - Define a game with $U_1(x_1) = U(x_1)$

$$U_s(x_s) = \hat{U}'(x) \cdot x_s, \ s \ge 2$$

  - At optimal, the efficiency is $U_1(c) = U(x^*)$
  - At equilibrium $\hat{U}_1'(x_1) = \hat{U}'(x_1) = \hat{U}_s'(x_s) = \hat{U}'(x)(1 - x_s / C)$

- Then, $x_1 \to x$ as the player number goes to infinity.
- Thus, the efficiency at equilibrium approaching

$$\hat{U}_1(x) + \hat{U}'(x)(C - x) = \hat{U}(x) + \hat{U}'(x)(x^* - x).$$

- In every resource allocation game, the price of anarchy is at least β.
  - Proof: let x^* and x are the optimal and equilibrium

$$\sum_s U_s(x_s^*) \le \sum_s \frac{i}{\beta}(U_s(x_s) + \hat{U}_s'(x_s)(x_s^* - x_s)) \le \frac{1}{\beta}\sum_s U_s(x_s)$$

- The bound $\beta = 3/4$.
  - Proof: setting $U(x) = x \,\&\, x = 1/2 \,\&\, c = x^* = 1$ shows the bound is at most 3/4.
  - Assume $x < x^* = c$, we have

$$
\begin{aligned}
U(x) + \hat{U}'(x)(x^* - x) &= U(x) + (1 - x/x^*)U'(x)(x^* - x) \\
&\geq U(x) + (1 - x/x^*)(U(x^*) - U(x)) \\
&= (x/x^*)U(x) + (1 - x/x^*)U(x^*) \\
&\geq (x/x^*)^2 U(x) + (1 - x/x^*)U(x^*) \\
&\geq \frac{3}{4}U(x^*).
\end{aligned}
$$

Network design games

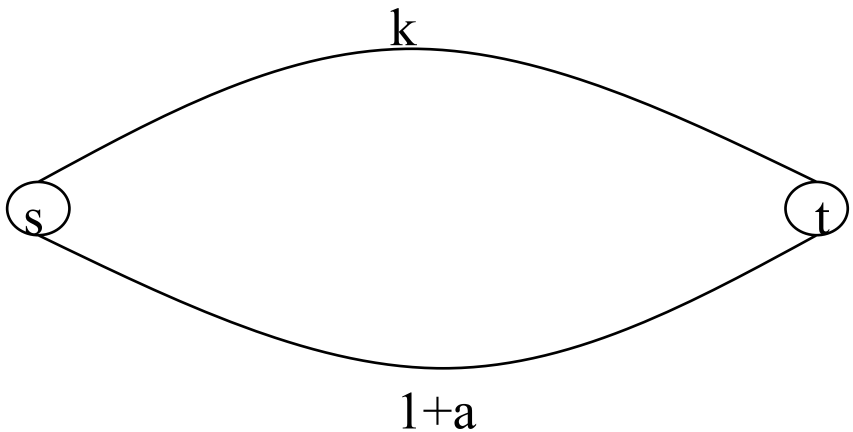
- Consider a network (V, E) with a nonnegative cost c_e for each edge $e \in E$
  - N source-destination pairs (players)
  - Each player i can choose a path $s_i \in P_i$
  - The total cost is $c(s) = \sum_{e \in \cup s_i} c_e$
  - Let n_e denote the number of players whose paths are using edge e. Each of those players pays a share $\pi_e = c_e / n_e$ of the cost
  - The cost for each player i is

$$c_i(s_i; s_{-i}) = \sum_{e \in s_i} c_e / n_e$$

- $\{V, E; c\}$ is a potential game with potential function

$$\Phi(s) = \sum_{e} \sum_{j=1}^{n_e} \frac{c_e}{j}.$$

- Every network design game has at least one Nash equilibrium.



- k players and $a > 0$ arbitrarily small
  - Two Nash equilibria: all chooses the upper edges, or all choose the lower edge

Price of stability

- Price of stability

$$\text{price of stability} = \frac{\text{obj fn value of the best equilibrium}}{\text{optimal obj fn value}}$$

- Since $C(s) \le \Phi(s) \le (1 + 1/2 + \cdots + 1/k)C(s)$, the price of stability is at most $1 + 1/2 + \cdots + 1/k$.