Optimization and Control of Networks

S-modular Games and Power Control

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Agenda

- S-modular games
  - Supermodular games
  - Submodular games
- Power control
  - Power control via pricing
  - A general framework for distributed power control

Supermodular games

- Characterized by "strategic complementarities"
- Supermodular games are remarkable
  - Pure strategy Nash equilibrium exists
  - The equilibrium set has an order structure with extreme elements
  - Many solution concepts yield the same prediction
  - Analytically appealing
    - Have nice comparative statics and behave well under various learning or adaptive algorithms
  - Encompass many applied models

Monotone comparative statics

- Def: suppose $X \subseteq R$ and T some partially ordered set. A function $f : X \times T \to R$ has increasing differences (supermodular) in (x,t) if for all $x' \geq x$ and $t' \geq t$,

$$f(x',t') - f(x,t') \geq f(x',t) - f(x,t).$$

- The incremental gain to choose a higher x is greater when t is higher.
- The increasing differences is symmetric, i.e., if $t' \geq t$, then $f(x,t') - f(x,t)$ is nondecreasing in x .

- Lemma: if f is twice continuously differentiable, then f has increasing differences iff $t' \geq t$ implies

$$f_x(x,t') \geq f_x(x,t)$$

for all x, or alternatively that, for all x, t,

$$f_{xt}(x,t) \geq 0$$

- A central question: when $x(t) = \arg\max_{x \in X} f(x,t)$ will be increasing in t ?
- Theorem (Topkis): Let $X \subseteq R$ be compact and T a partially ordered set. Suppose $f : X \times T \to R$ has increasing differences in (x,t), and is upper semi-continuous in x . Then,
  - For all t, $x(t)$ exists and has a greatest and least element $\bar{x}(t)$ and $\underline{x}(t)$.
  - $\bar{x}(t)$ and $\underline{x}(t)$ are increasing in t .

- Proof:
  - Existence: X is compact and f is upper semicontinuous
  - Take a sequence $\{x^k\}$ in $x(t)$. From compactness, there exists a limit point $\bar{x} = \lim_{k\to\infty} x^k$. Then for all x,

    $$f(x^k,t) \geq f(x,t) \Rightarrow f(\bar{x},t) \geq f(x,t).$$

    Thus, $\bar{x} \in x(t)$ and $x(t)$ is therefore closed. It follows that $x(t)$ has a greatest and least element.
  - Let $x \in x(t)$ and $x' \in x(t')$. Then, $f(x,t) - f(\min(x,x'),t) \geq 0$, which implies $f(\max(x,x'),t) - f(x',t) \geq 0$. By the increasing difference, $f(\max(x,x'),t') - f(x',t') \geq 0$. Thus $\max(x,x')$ maximizes $f(\cdot,t')$. Now, pick $x = \bar{x}(t)$ and $x' = \bar{x}(t')$, it follows that $x' \geq x$. A similar argument applies to $\underline{x}(t)$.

Supermodular games

- Def: the game $G = \{N, S_{i\in N}, u_{i\in N}\}$ is a super-modular game if for all i,
  - S_i is a compact subset of R
  - u_i is upper semicontinuous in s_i, s_{-i}
  - u_i has increasing differences in (s_i, s_{-i})
- Corollary: suppose $G = \{N, S_{i\in N}, u_{i\in N}\}$ is a super-modular game. Define the best response function $B_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$. Then
  - $B_i(s_{-i})$ has greatest and least element $\overline{B}_i(s_{-i})$ and $\underline{B}_i(s_{-i})$
  - $\overline{B}_i(s_{-i})$ and $\underline{B}_i(s_{-i})$ are increasing in s_{-i}

Example: Bertrand game

- Two firms: form 1 and firm 2 with prices $p_1, p_2 \in [0,1]$
- Payoff $u_i(p_i, p_j) = p_i(1 - 2p_i + p_j)$
- It is a supermodular game, since $\frac{\partial^2 u_i}{\partial p_i \partial p_j} > 0$.
- Solve by iterated strict dominance
  - Let $S_i^0 = [0,1]$, then $S_i^1 = [1/4, 1/2]$.
    - If $p_i < 1/4$, then $\frac{\partial u_i}{\partial p_i} > 1 - 4 \cdot \frac{1}{4} + p_j \geq 0 \Rightarrow p_i < \frac{1}{4}$ is strictly dominated.
    - If $p_i > 1/2$, then $\frac{\partial u_i}{\partial p_i} < 1 - 4 \cdot \frac{1}{2} + p_j \leq 0 \Rightarrow p_i > \frac{1}{2}$ is strictly dominated.
  - Let $S_i^k = [\underline{s}^k, \bar{s}^k]$, then

$$\underline{s}^k = 1/4 + \underline{s}^{k-1}/4 = 1/4 + 1/16 + \underline{s}^{k-2}/16 = \cdots = 1/4 + \cdots + 1/4^k + \underline{s}^0/4^k$$

$$\bar{s}^k = 1/4 + \bar{s}^{k-1}/4 = 1/4 + 1/16 + \bar{s}^{k-2}/16 = \cdots = 1/4 + \cdots + 1/4^k + \bar{s}^0/4^k$$

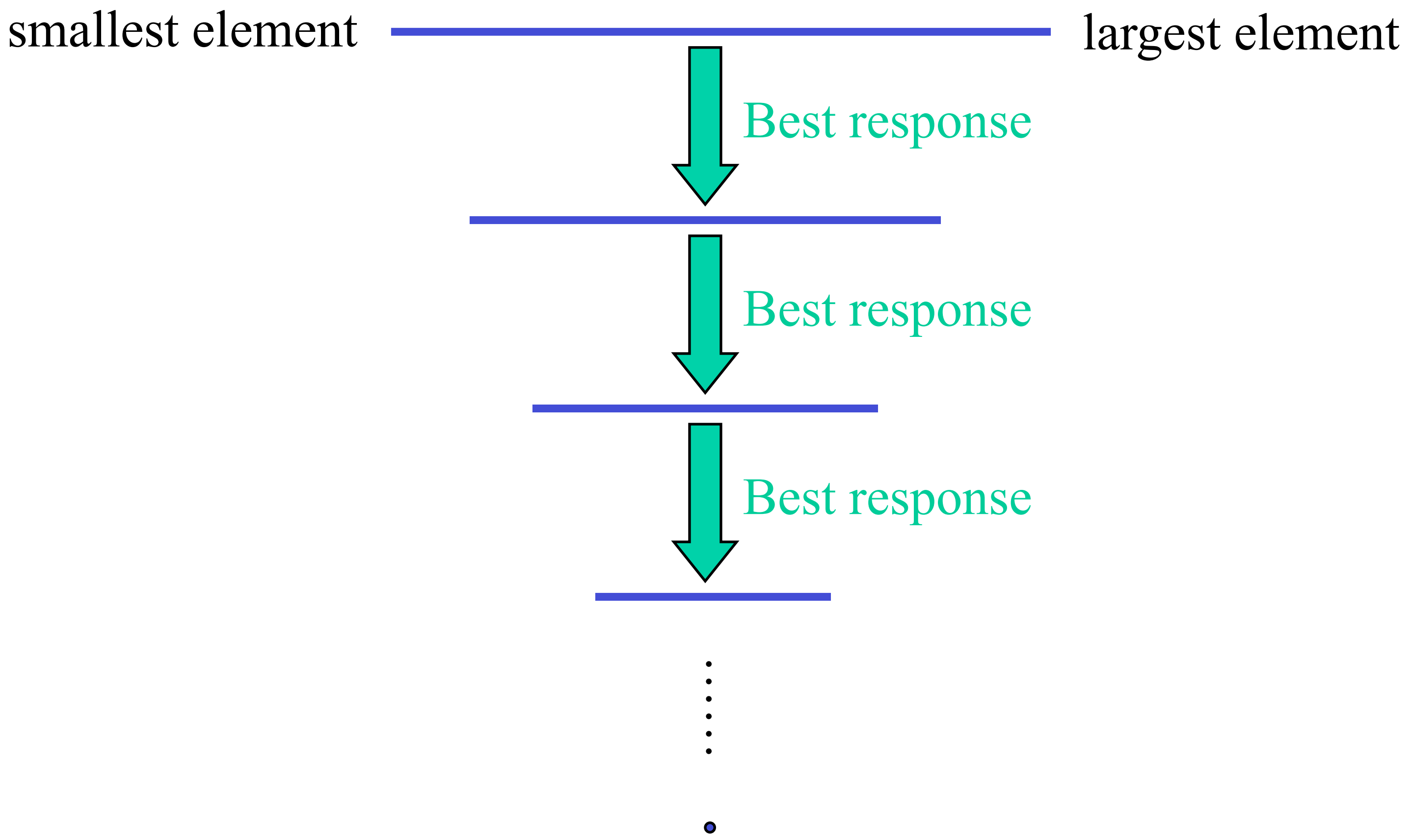
  - (1/3, 1/3) is the only Nash equilibrium.

- Theorem: let $G = \{N, S_{i \in N}, u_{i \in N}\}$ be a supermodular game. Then the set of strategies surviving iterated strict dominance (ISD) has greatest and least element $\bar{s}$ and $\underline{s}$, which are pure strategy Nash equilibria.
- Corollary:
  - Pure Nash equilibrium exists.
  - The largest and smallest strategies compatible with ISD, rationalizability, correlated equilibrium and Nash equilibrium are the same.
  - If a supermodular game has a unique Nash Equilibrium, it is dominance solvable.

- Proof: let $S^0 = S$ and $s^0 = (s_1^0, \cdots, s_{|N|}^0)$ be the largest element of S. Let $s_i^1 = \overline{B}_i(s_{-i}^0)$ and $S_i^1 = \{s_i \in S_i^0 : s_i \le s_i^1\}$. If $s_i \notin S_i^1$, i.e., $s_i > s_i^1$, then it is dominated by s_i^1.
  - By increasing differences
    $$u_i(s_i, s_{-i}) - u_i(s_i^1, s_{-i}) \le u_i(s_i, s_{-i}^0) - u_i(s_i^1, s_{-i}^0) < 0$$
  - Also note that $s^1 \le s^0$
- Iterate and define $s_i^k = \overline{B}_i(s_{-i}^{k-1})$ and $S_i^k = \{s_i \in S_i^{k-1} : s_i \le s_i^k\}$. Now if $s^k \le s^{k-1}$, then $s_i^{k+1} = \overline{B}_i(s_{-i}^k) \le \overline{B}_i(s_{-i}^{k-1}) = s_i^k$. So, $\{s^k\}$ is a decreasing sequence and has a limit denoted by $\bar{s}$. Only the strategies $s_i \le \bar{s}_i$ are undominated.

- Similarly, start with $s^0 = (s_1^0, \cdots, s_{|N|}^0)$ the smallest element of S and identify $\underline{s}$.
- Show $\bar{s}$ and $\underline{s}$ are Nash equilibria.
  - For all i and s_i, $u_i(s_i^{k+1}, s_{-i}^k) \geq u_i(s_i, s_{-i}^k)$
  - Take the limit as $k \to \infty$, $u_i(\bar{s}_i, \bar{s}_{-i}) \geq u_i(s_i, \bar{s}_{-i})$.
  - Similarly, prove $\underline{s}$ is a Nash equilibrium

Illustrative diagram



Submodular games

- Def: suppose $X \subseteq R$ and T some partially ordered set. A function $f: X \times T \rightarrow R$ has decreasing differences (submodular) in (x,t) if for all $x' \geq x$ and $t' \geq t$,

$$f(x',t') - f(x,t') \leq f(x',t) - f(x,t).$$

- A game is a submodular game if the payoff functions are submodular.
- More generalizations

Monotonicity

- Def: let A and B are two sets. We say $A \prec B$ if for any $a \in A$ and $b \in B$, $\min(a,b) \in A$ and $\max(a,b) \in B$.
  - Component-wise operations
- For constraint sets $S_i(s_{-i}) \subseteq S_i$, if

$$s_{-i} \le s'_{-i} \Rightarrow S_i(s'_{-i}) \prec S_i(s_{-i}),$$

  then the set S_i possess the descending property. The ascending property can be defined when the relation is reversed.

- Theorem: for a submodular game with descending $S_i(\cdot)$,
  - An Nash equilibrium exists.
  - The best response strategy

$$B_i(s_{-i}) = \min\{\arg \min_{s_i \in S_i(s_{-i})} u_i(s_i, s_{-i})\}$$

    monotonically converges to an equilibrium.
- Proof: Follows monotonicity of the best response. Similar to the proof of former theorem.
- Similar result exists for a supermodular game with ascending $S_i(\cdot)$.

Power control

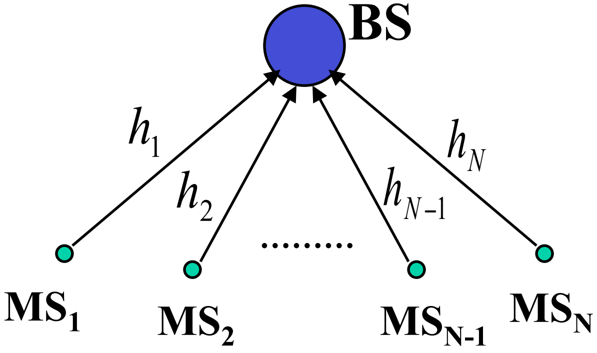
- An important component of radio resource management
  - Meet target BER or SIR while limiting interference
  - Increase capacity by minimizing interference
  - Extend battery life
- Users assigned utilities that are functions of the power they consume and the signal-to-interference ratio (SIR) they attain
- Try to find a good balance between high SIR (or meeting target SIR) and low power consumption

Power control via pricing

- Consider a single-cell network with a set N of users at uplink
- Each user i can choose a power $p_i \in [p_i^{\min}, p_i^{\max}]$
- The SIR for user i

$$\gamma_i = \frac{h_i p_i}{\sum_{j \neq i} h_j p_j + \sigma^2}$$

where h_i is the channel gain from MS to BS and σ^2 is the noise variance.



- Consider payoff $u_i(p_i, p_{-i}) = f(\gamma_i) - \alpha_i p_i$
- $f(\cdot)$ assumed to be increasing
- When the utilities are supermodular?

$$\frac{\partial u_i(p_i, p_{-i})}{\partial p_i} = f'(\gamma_i)\frac{\gamma_i}{p_i} - c$$

$$\frac{\partial^2 u_i(p_i, p_{-i})}{\partial p_i \partial p_j} = -\frac{\gamma_i^2 h_j}{h_i p_i^2}(\gamma_i f''(\gamma_i) + f'(\gamma_i)),\ \ j \neq i$$

- Requires $\gamma_i f''(\gamma_i) + f'(\gamma_i) < 0$
  - Example: some concave functions

- Power control algorithm
  - At time $t = 0$, let $p(0) = p^{\min}$.
  - At each time $t = k$, set user i power

$$p_i(k) = \min\{\arg\max_{p_i} u_i(p_i, p_{-i}(k-1))\}$$

- The above algorithm converges to a Nash equilibrium that is the smallest equilibrium.

A general framework for distributed power control

- Consider a set N of users and a set of M base stations
- User j uses power p_j
- Denote by h_{kj} the gain of user j at base station k
- The SIR of user j at base station k is $p_j \mu_{kj}$ with

$$\mu_{kj} = \frac{h_{kj}}{\sum_{i \neq j} p_i h_{ki} + \sigma_k^2}$$

Different power control schemes

- Fixed assignment: the user j is assigned to BS a_j with a SIR requirement γ_j. The constraints is

$$p_j \geq I^{FA}(p) = \frac{\gamma_i}{\mu_{a_j j}(p)}$$

- Minimum power assignment, limited diversity and multiple reception are have the constraints of the same form

$$p_j \geq I(p)$$

Standard interference function

- The standard interference function $I(p)$ satisfies the following properties
  - Positivity: $I(p) > 0$
  - Monotonicity: if $p \geq p' \Rightarrow I(p) \geq I(p')$
  - Scalability: for $a > 1$, $aI(p) \geq I(ap)$

- Define a submodular game
  - Payoff $u_j(p) = p_j$
  - Constraint set $S_j(p_{-j}) = \{p_j : p_j \geq I_j(p),\ 0 \leq p_j \leq p'_j\}$

    with p' a feasible solution to $p \geq I(p)$

- Theorem: if a feasible solution p' exists, then
  - There is a fixed point to equation $p = I(p)$
  - The best response strategy converges to an equilibrium.