Economic equilibrium and optimization problems using GAMS
Notes 4: Imperfect competition and games

James R. Markusen
University of Colorado, Boulder

Monopoly
Cournot duopoly
Oligopoly with free entry and exit
Nash equilibrium with discrete strategies
M4-1 Simple partial-equilibrium monopoly model

Revenue: price times quantity, but now price is a function of quantity: \( p = p(X) \), revenue = \( p(X)X \)

\[
\text{Marginal revenue} = \frac{d(p(X)X)}{dX} = p + X \frac{dp}{dX}
\]

\[
MR = p + p \frac{X \, dp}{p \, dX} = p \left( 1 - \frac{1}{\eta} \right)
\]

\[
\eta = -\frac{p \, dX}{X \, dp} \quad \text{is the price elasticity of demand}
\]
Profits = Revenue - Cost = \( p(X)X - cX \)

First-order condition for profit max: \( MR = MC \)

\[
\max \Pi \quad \Rightarrow \quad p \left(1 - \frac{1}{\eta}\right) = c
\]

Special case: demand given by

\[
X = p^{-\sigma}M \quad \text{where M is income } \sigma > 1
\]

\[
\frac{dX}{dp} = -\sigma p^{-\sigma-1}M \quad \quad \frac{p}{X} \frac{dX}{dp} = -\sigma
\]

\[
MR = MC \quad \Rightarrow \quad p \left(1 - \frac{1}{\sigma}\right) = c
\]
$TITLE: M4-1 simple partial equilibrium monopoly market
* constant price elasticity of demand function gives simple markup rule
* MK = 1/SI where SI (sigma in the notes) is the price elasticity of demand

PARAMETERS
   SI   price elasticity of demand
   M    income
   C    marginal cost (constant);

   SI = 5;
   M = 10;
   C = 1;

VARIABLES
   PR   profit;

NONNEGATIVE VARIABLES
   X    output or demand
   P    price
   MK   markup;

EQUATIONS
   PROFIT   profit
   DEMAND   supply = demand
   FOC1     marginal cost >= marginal revenue using SI
   FOC2     marginal cost >= marginal revenue using variable MK
   MARKUP   markup formula;
PROFIT..  PR =E= P*X - C*X;

DEMAND..  X =E= P**(-SI)*M;

FOC1..    C =G= P*(1-1/SI);

FOC2..    C =G= P*(1-MK);

MARKUP..  MK =G= 1/SI;

PR.L = 1;
P.L = 1.25;
X.L = 3;

MODEL PMAXNLP /PROFIT, DEMAND/;
SOLVE PMAXNLP MAXIMIZING PR USING NLP;

MODEL PMAXMCP1 /FOC1.X, DEMAND.P/;
SOLVE PMAXMCP1 USING MCP;

MODEL PMAXMCP2 /FOC2.X, DEMAND.P, MARKUP.MK /;
SOLVE PMAXMCP2 USING MCP;

PARAMETERS
CSMONO consumer surplus under monopoly
PRMONO profits under monopoly
WMONO welfare under monopoly
CSCOMP consumer surplus under competition
PRCOMP profits under competition
WCOMP welfare under competition;
CSMONO = 1/(SI-1)*P.L*X.L;
PRMONO = P.L*X.L - C*X.L;
WMONO = CSMONO + PRMONO;

* compare to the competitive solution by constraining MK = 0;
MK.FX = 0;

SOLVE PMAXMCP2 USING MCP;
CSCOMP = 1/(SI-1)*P.L*X.L;
PRCOMP = P.L*X.L - C*X.L;
WCOMP = CSCOMP + PRCOMP;

DISPLAY CSMONO, PRMONO, WMONO;
DISPLAY CSCOMP, PRCOMP, WCOMP;

example showing point from economics of regulation
Suppose that there is a fixed cost to the firm FC
Then the competitive solution means that the firm is making losses
but the competitive solution is still socially optimal
First best policy is marginal cost pricing with a subsidy

PARAMETERS
FC fixed cost /0.5/;
MK.UP = +INF;
MK.LO = 0;

SOLVE PMAXMCP2 USING MCP;

CSMONO = 1/(SI-1)*P.L*X.L;
PRMONO = P.L*X.L - C*X.L - FC;
WMONO = CSMONO + PRMONO;

* compare to the competitive solution by constraining MK = 0;

MK.FX = 0;

SOLVE PMAXMCP2 USING MCP;

CSCOMP = 1/(SI-1)*P.L*X.L;
PRCOMP = P.L*X.L - C*X.L - FC;
WCOMP = CSCOMP + PRCOMP;

DISPLAY CSMONO, PRMONO, WMONO;
DISPLAY CSCOMP, PRCOMP, WCOMP;
M4-2 Partial-equilibrium oligopoly model with free entry and exit

Firms have a cost function that has a constant marginal cost $c$ and a fixed cost $f$.

Marginal cost in units of labor is denoted by $mc$ and total cost ($tc$) and average cost ($ac$) for an $X$ firm are as follows:

\[
\begin{align*}
tc &= cX + f \\
ac &= \frac{tc}{X} = c + \frac{f}{X} \\
mc &= c
\end{align*}
\]

Auto industry: Minimum efficient scale, thousands of units per year

<table>
<thead>
<tr>
<th></th>
<th>1500</th>
<th>1000</th>
<th>500</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>Foundry</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pressing</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Powertrain</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Final assembly</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[ ac = mc + \frac{fc}{X} \]
Cost penalties from sub-optimal scale

<table>
<thead>
<tr>
<th>Level of production</th>
<th>50,000</th>
<th>100,000</th>
<th>200,000</th>
<th>400,000</th>
<th>800,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost penalty (%)</td>
<td>20</td>
<td>10-15</td>
<td>3-5</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Size of plant (% of MES)</th>
<th>100</th>
<th>80</th>
<th>60</th>
<th>30</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost penalty (%)</td>
<td>0</td>
<td>3</td>
<td>6.8</td>
<td>19.5</td>
<td>34.5</td>
</tr>
</tbody>
</table>

Cournot-Nash (or Cournot for short) competition in which firms pick a quantity as a best response to their rivals’ quantities.

Revenue for a Cournot firm i and selling in country j is given by the price times quantity of the firm’s sales. Price is a function of all firms’ sales.
US automobile production 2005  (excludes firms exclusively producing trucks)

<table>
<thead>
<tr>
<th>Number</th>
<th>Market share</th>
<th>Cumulative market share</th>
</tr>
</thead>
<tbody>
<tr>
<td>3382315</td>
<td>0.288</td>
<td>0.288</td>
</tr>
<tr>
<td>2965872</td>
<td>0.252</td>
<td>0.540</td>
</tr>
<tr>
<td>1652703</td>
<td>0.141</td>
<td>0.681</td>
</tr>
<tr>
<td>1283829</td>
<td>0.109</td>
<td>0.790</td>
</tr>
<tr>
<td>973290</td>
<td>0.083</td>
<td>0.873</td>
</tr>
<tr>
<td>835946</td>
<td>0.071</td>
<td>0.944</td>
</tr>
<tr>
<td>251147</td>
<td>0.021</td>
<td>0.965</td>
</tr>
<tr>
<td>125086</td>
<td>0.011</td>
<td>0.976</td>
</tr>
<tr>
<td>122328</td>
<td>0.010</td>
<td>0.986</td>
</tr>
<tr>
<td>75200</td>
<td>0.006</td>
<td>0.993</td>
</tr>
<tr>
<td>88003</td>
<td>0.007</td>
<td>1.000</td>
</tr>
</tbody>
</table>

11755719
\[ R_i = p(X)X_i, \quad \text{X is total sales: } X = \sum_i X_i \]

Cournot conjectures imply that \( \frac{\partial X}{\partial X_i} = 1 \); a one-unit increase in the firm’s own supply is a one-unit increase in market supply.

Marginal revenue is then

\[
\frac{\partial R_i}{\partial X_i} = p + X_i \frac{\partial p}{\partial X} \frac{\partial X}{\partial X_i} = p + X_i \frac{\partial p}{\partial X} \quad \text{since} \quad \frac{\partial X}{\partial X_i} = 1
\]

Now multiple and divide the right-hand equation by total market supply and also by the price.

\[
\frac{\partial R_i}{\partial X_i} = p + X_i \frac{\partial p}{\partial X} = p + p \frac{X_i}{X} \left[ \frac{X \partial p}{p \partial X} \right]
\]
The term in square brackets in (11.6) is just the inverse of the price elasticity of demand.

\[
\frac{\partial R_i}{\partial X_i} = p \left[ 1 - \frac{X_i}{X} \frac{1}{\sigma} \right] \quad \sigma = -\left[ \frac{p}{X} \frac{\partial X}{\partial p} \right] \quad \text{(elasticity of demand)}
\]

The term \(X_i/X_j\) in (11.6) is just firm i’s market share in market j, which we can denote by \(s_{ij}\).

\[
mr_i = p \left[ 1 - \frac{s_i}{\sigma} \right] = mc_i \quad mr_i = p \left[ 1 - \frac{1}{N\sigma} \right] = mc_i
\]

If all firms are identical, then each firm’s market share is just 1/N where N is the number of firms in equilibrium.
NONNEGATIVE VARIABLES
    X    output or demand
    P    price
    MK   markup
    N    number of firms in equilibrium;

EQUATIONS
    DEMAND     supply = demand
    PRICING    marginal cost >= marginal revenue using variable MK
    MARKUP     markup formula
    ZEROPROF   zero profits;

    DEMAND..   N*X =E= P**(-SI)*M;

    PRICING..  C =G= P*(1-MK);

    MARKUP..   MK =G= 1/(N*SI);

    ZEROPROF.. 0 =G= P*X - C*X - FC;

P.L = 1; X.L = 10; N.L = 2.5; MK.L = 1/(N.L*SI);

MODEL FREEENT /DEMAND.P, PRICING.X, MARKUP.MK, ZEROPROF.N /;
SOLVE FREEENT USING MCP;
Counterfactual: double the size of the economy: $M = 50$.

This creates a welfare gain (per capita) that would not be present in a competitive model or the monopoly model.

- output per firm rises, firm’s become more efficient
- thus average cost = price falls, which is a measure of efficiency or productivity
- the markup rate falls, indicating a smaller difference between price and marginal cost ($p = mc$ is required for first best)
Cournot and Bertrand oligopoly with continuous strategies

Two firms h and f (as in countries h and f) produce imperfect substitutes for the world market:

(a) linear inverse demand curve for each good
(b) each firm has a constant marginal cost
(c) fixed costs are ignored.

\[ p_h = \alpha - \beta X_h - \gamma X_f \quad p_f = \alpha - \beta X_f - \gamma X_h \quad \beta \geq \gamma \]

\[ \pi_i = p_h X_i - c_h X_i = (\alpha - \beta X_i - \gamma X_j) X_i - c_h X_i \quad i \neq j \]

Cournot Nash competition is the behavioral assumption that each firm maximizes its profits treating their rival’s output as fixed. (Best response.)
Best response Cournot-Nash equilibrium is the solution to the two first-order conditions for \( h \) and \( f \)

\[
    c_h \geq \alpha - 2\beta X_h - \gamma X_f \quad \quad \quad c_f \geq \alpha - 2\beta X_f - \gamma X_h
\]

These FOC are commonly referred to as “best response” or “reaction” functions. Here they can be rewritten as:

\[
    X_h = \frac{(\alpha - c_h)}{2\beta} - \frac{\gamma}{2\beta} X_f \quad \quad \quad X_f = \frac{(\alpha - c_f)}{2\beta} - \frac{\gamma}{2\beta} X_h
\]

They can be solved explicitly, easy in symmetric case with identical marginal costs (the two outputs are then identical):

\[
    X_i = \frac{\alpha - c}{2\beta + \gamma}
\]
$TITLE: M4-3 James Markusen, University of Colorado, Boulder  
* Cournot with continuous strategies

$ONTEXT

begin with Cournot douopoly

single unified market, constant marginal costs

goods XH and XF are imperfect substitutes

inverse demand functions \( PH = \alpha - \beta XH - \gamma XF \quad BETA > \gamma GAMMA \)

maximizing profits gives FOC (implicity reaction functions)

\[
\text{PROFIT} = PH*XH - CH*XH = (\alpha - \beta XH - \gamma XF)*XH - CH*XH
\]

first order condition: \( \alpha - 2\beta XH - \gamma XF - CH = 0 \)

$OFFTEXT

PARAMETERS

\begin{itemize}
  \item \text{ALPHA} \quad \text{intercept of demand curve}
  \item \text{BETA} \quad \text{slope of inverse demand curve wrt own quantity}
  \item \text{GAMMA} \quad \text{slope of inverse demand curve wrt rival's quantity}
  \item \text{CH} \quad \text{marginal cost of home firm}
  \item \text{CF} \quad \text{marginal cost of foreign firm}
\end{itemize}

RESULTS(*,*)

\begin{itemize}
  \item \text{ALPHA} = 12;
  \item \text{BETA} = 2;
  \item \text{GAMMA} = 1.5;
  \item \text{CH} = 1;
  \item \text{CF} = 1;
\end{itemize}
VARIABLES
PROFH  profit of firm h
PROFF  profit of firm f;

NONNEGATIVE VARIABLES
PH  price of XH
PF  price of XF
XH  quantity of XH
XF  quantity of XF;

EQUATIONS
PROFIT H  profit of firm h
PROFIT F  profit of firm f
PRICE H  inverse demand curve facing firm h
PRICE F  inverse demand curve facing firm f
HCOURNOT  cournot FOC for firm h (reaction function)
FCOURNOT  cournot FOC for firm f (reaction function);

PROFIT H..  PROFH =E= PH*XH - CH*XH;

PROFIT F..  PROFF =E= PF*XF - CF*XF;

PRICE H..  PH =E= ALPHA - BETA*XH - GAMMA*XF;

PRICE F..  PF =E= ALPHA - BETA*XF - GAMMA*XH;

HCOURNOT..  CH =G= ALPHA - 2*BETA*XH - GAMMA*XF;

FCOURNOT..  CF =G= ALPHA - 2*BETA*XF - GAMMA*XH;
* actually only need the two first-order conditions to solve and then
* back out other variables, but harmless to include the other variables

```gams
MODEL COURNOT /HCOURNOT.XH, FCOURNOT.XF,
                 priceh.ph, pricef.pf, profith.profh, profitf.proff/;
SOLVE COURNOT USING MCP;
```

```gams
RESULTS("PROFH", "COURNOT") = PROFH.L;
RESULTS("PROFF", "COURNOT") = PROFF.L;
RESULTS("XH", "COURNOT") = XH.L;
RESULTS("XF", "COURNOT") = XF.L;
RESULTS("PH", "COURNOT") = PH.L;
RESULTS("PF", "COURNOT") = PF.L;
```

* solve for Cournot equilibrium using nlp via "diagonalization"
* max profits for H holding XF constant
* free up XF, hold XH at its solution value, max profits for F
* free up XH, hold XF at its solution value, max profits for H
* repeat

```gams
SETS I /I1*I10/;
MODEL COURNOTNLP /PROFITH, PROFITF, PRICEH, PRICEF/;
XH.L = 1; XF.L = 1; PH.L = 1; PF.L = 1;
```
LOOP (I,

XH.UP = +INF; XH.LO = 0;
XF.FX = XF.L;

SOLVE COURNOTNLP USING NLP MAXIMIZING PROFH;

XF.UP = +INF; XF.LO = 0;
XH.FX = XH.L;

SOLVE COURNOTNLP USING NLP MAXIMIZING PROFF;

);

* solve for collusive outcome
XH.UP = +INF; XH.LO = 0; XF.UP = +INF; XF.LO = 0;

VARIABLES
    JPROF joint profits payoff;
EQUATIONS
    JPROFIT joint profit function;

JPROFIT.. JPROF =E= PROFH + PROFF;

MODEL COLLUSION /JPROFIT, PROFITH, PROFITF, PRICEH, PRICEF/;

SOLVE COLLUSION USING NLP MAXIMIZING JPROF;

RESULTS("PROFH", "JMAX") = PROFH.L;
RESULTS("PROFF", "JMAX") = PROFF.L;
RESULTS("XH", "JMAX") = XH.L;
RESULTS("XF", "JMAX") = XF.L;
RESULTS("PH", "JMAX") = PH.L;
RESULTS("PF", "JMAX") = PF.L;

* solve for the competitive outcome
* add two equations for price equals marginal cost, drop reaction functions

EQUATIONS
    COMPH   price equals marginal cost for XH
    COMPF   price equals marginal cost for XF;

    COMPH..  CH =G= PH;
    COMPF..  CF =G= PF;

MODEL  COMP /PROFITH.PROFH, PROFITF.PROFF, PRICEH.XH, PRICEF.XF,
            COMPH.PH, COMPF.PF/;

SOLVE  COMP USING MCP;

RESULTS("PROFH", "COMP") = PROFH.L;
RESULTS("PROFF", "COMP") = PROFF.L;
RESULTS("XH", "COMP") = XH.L;
RESULTS("XF", "COMP") = XF.L;
RESULTS("PH", "COMP") = PH.L;
RESULTS("PF", "COMP") = PF.L;

DISPLAY  RESULTS;
IMPORTANCE OF PHARMACEUTICAL R&D

In 2011 the pharmaceutical industry invested about €29,200 million in R&D in Europe. A decade of strong US market dominance led to a shift of economic and research activity towards the US from 1995–2005. Additionally, Europe is now facing increasing competition from emerging economies: rapid growth in the market and research environments in countries such as Brazil and China are contributing to the move of economic and research activities to non-European markets. The geographical balance of the pharmaceutical market – and ultimately the R&D base – is likely to shift gradually towards emerging economies.

ESTIMATED FULL COST OF BRINGING A NEW CHEMICAL OR BIOLOGICAL ENTITY TO MARKET ($ MILLION - YEAR 2011 $)


PHARMACEUTICAL R&D EXPENDITURE - ANNUAL GROWTH RATE (%)
4.4 Nash equilibria with discrete strategies

Gams has some great set features that allow a modeler to capture lots of very interesting economics.

Here, I present a simple example of a two-player normal-form game in which each player has three strategies.

This particular version is motivated by a two-country trade model with multinational firms in which there is one firm in each country. Each firm may:

- not enter, strategy 0
- enter with a single plant at home, exporting to the other country, strategy 1
- enter with plants in both countries, serving each market from a local plant, strategy 2
In an actual model, the numerical values in the payoff matrices are solved for from the underlying duopoly problem. Here I’ll just make up number consistent with the underlying example.

SETS R strategies for firm h /SH0, SH1, SH2/
   C strategies for firm f /SF0, SF1, SF2/

ALIAS(R,RR)
ALIAS(C,CC);

TABLE PAYOFFH(*,*)
   SF0   SF1   SF2
SH0   -.1   -.1   -.1
SH1   10     6     3
SH2   12     5     2 ;

TABLE PAYOFFF(*,*)
   SF0   SF1   SF2
SH0   -.1    10    12
SH1   -.1     6     5
SH2   -.1     3     2 ;
A best response Nash equilibrium, involves player h picking the row element that is the largest given the column selected by player f and vice versa (f picks the highest column for h’s row pick).

There is GAMS command that identifies the best response strategy. First, some GAMS notation.

\[ X = 1 \text{($Y$ EQ 1)} \] means:

“set X equal to one if Y is equal to one, otherwise set \( X = 0 \)"

\[ X\text{($Y$ EQ 1)} = 1 \] means something subtly different:

“set X equal to one if Y is equal to one, otherwise leave the existing value of X unchanged”

We will need the first version here.
Let \( \text{ROWMAX}(R,C) \) be a matrix for \( h \).

A value = 1 in cell \((R,C)\) will denote \( h \)'s best response row \( R \) when \( f \) plays column strategy \( C \). Non-optimal responses = 0.

Let \( \text{COLMAX}(R,C) \) be a matrix for \( f \).

A value = 1 in cell \((R,C)\) will denote \( f \)'s best response column \( C \) when \( h \) plays row strategy \( R \). Non-optimal responses = 0.

The crucial GAMS command is \( \text{SMAX} \) (set max):

\[ \text{SMAX}(RR, \text{PAYOFFH}(RR,C)) \] is the maximum value of the parameter \( \text{PAYOFFH} \) over the rows, for a given column \( C \).
The best-response matrices with zeros and ones are given by:

\[
\text{ROWMAX}(R, C) = 1 \times \left( \text{PAYOFFH}(R, C) = \text{SMAX}(RR, \text{PAYOFFH}(RR, C)) \right);
\]

\[
\text{COLMAX}(R, C) = 1 \times \left( \text{PAYOFFF}(R, C) = \text{SMAX}(CC, \text{PAYOFFF}(R, CC)) \right);
\]

Now multiple these two matrices together element by element, to get a new matrix \( \text{NASHEQ}(R, C) \).

A one denote a best response for both \( h \) and \( f \), and hence that \( (R, C) \) cell is a Nash equilibrium.

\[
\text{NASHEQ}(R, C) = \text{ROWMAX}(R, C) \times \text{COLMAX}(R, C);
\]
Finally, the profits at each Nash equilibrium are given by

\[
\text{PROFHNE}(R, C) = \text{PAYOFFH}(R, C) \times \text{NASHEQ}(R, C);
\]

\[
\text{PROFFNE}(R, C) = \text{PAYOFFF}(R, C) \times \text{NASHEQ}(R, C);
\]

This technique will find ALL pure-strategy Nash equilibria. The second example shows a case of multiple equilibria.

Case 1: each firm chooses one plant and exports to the other country (1,1), exporting duopoly shown above

Case 2: three equilibria. Exporting duopoly as in Case 1, or one firm chooses two plants, and the other firm does not enter: (1,1), (2,0), (0,2)

Case 3: each firm chooses two plants, a horizontal multinational duopoly (2,2)
4.5 Networks and logistics

This is a proto-typical model of an common operations research problem.

In this example, there are three production plants and three markets.

(plant locations and markets are distinct, but that is not important to the problem)

SETS
I plants /GUANGDONG, HERMOSILLO, BILOXI/
J markets /NEW-YORK, CHICAGO, DENVER/;

In the first simple example, plants have fixed capacity and markets have a fixed demand (capacity must be GE to demand or GAMS returns “infeasible” as a solution.)
PARAMETERS
A(I)    plant capacity /GUANGDONG 4, HERMOSILLO 3, BILOXI 4/
B(J)    market size j /NEW-YORK 3, CHICAGO 2, DENVER 1/
C(I)    plant marginal cost of production /GUANGDONG 1, HERMOSILLO 1, BILOXI 3/
T(I,J)  transport cost rate from market i to j
F       freight rate parameter /90/;

Distance between plants and markets is crucial. Here is how to declare and assign a two-dimension parameter in GAMS.

TABLE DIST(I,J)  distance
    NEW-YORK  CHICAGO  DENVER
GUANGDONG      9        8        7
HERMOSILLO     4        2        1.5
BILOXI         2        2        3;

The following allows distance to be converted to costs.

T(I,J) = F*DIST(I,J)/500;
Here are the variables and equations.

VARIABLES
   COST variable cost to be minimized;
   
NONNEGATIVE VARIABLES
   X(I,J) shipment from i to j;
   
EQUATIONS
   SUPPLY(I) supply constraint
   DEMAND(J) demand constraint
   OBJDEF objective function to be minimized;
   
SUPPLY(I).. A(I) =G= SUM(J, X(I,J));

DEMAND(J).. SUM(I, X(I,J)) =G= B(J);

OBJDEF.. COST =E= SUM((I,J), X(I,J)*(C(I) + T(I,J)));

MODEL MNLP /SUPPLY, DEMAND, OBJDEF/;
   X.L(I,J) = 1;
SOLVE MNLP USING NLP MINIMIZING COST;
---- VAR X shipment from i to j

<table>
<thead>
<tr>
<th></th>
<th>LOWER</th>
<th>LEVEL</th>
<th>UPPER</th>
<th>MARGINAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>GUANGDONG .NEW-YORK</td>
<td>.</td>
<td>3.000</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>GUANGDONG .CHICAGO</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>0.180</td>
</tr>
<tr>
<td>GUANGDONG .DENVER</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>0.090</td>
</tr>
<tr>
<td>HERMOSILLO .NEW-YORK</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>HERMOSILLO .CHICAGO</td>
<td>.</td>
<td>2.000</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>HERMOSILLO .DENVER</td>
<td>.</td>
<td>1.000</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>BILOXI .NEW-YORK</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>0.740</td>
</tr>
<tr>
<td>BILOXI .CHICAGO</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>1.100</td>
</tr>
<tr>
<td>BILOXI .DENVER</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>1.370</td>
</tr>
</tbody>
</table>

Interpretation of the marginal *0.740* in an NLP program

Cost of sending 1 unit from Biloxi to NY: *0.360 + 3.000 = 3.360*
Saving from not sending 1 unit from Guangdong to NY: *1.620 + 1 = 2.620*.

*3.360 - 2.620 = 0.740* increase in total cost of serving NY.
Counterfactual: raise demand in Chicago to 5.

B("CHICAGO") = 5;

---- VAR X  shipment from i to j

<table>
<thead>
<tr>
<th></th>
<th>LOWER</th>
<th>LEVEL</th>
<th>UPPER</th>
<th>MARGINAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>GUANGDONG .NEW-YORK</td>
<td>.</td>
<td>1.000</td>
<td>+INF</td>
<td></td>
</tr>
<tr>
<td>GUANGDONG .CHICAGO</td>
<td>.</td>
<td>2.000</td>
<td>+INF</td>
<td></td>
</tr>
<tr>
<td>GUANGDONG .DENVER</td>
<td>.</td>
<td>1.000</td>
<td>+INF</td>
<td></td>
</tr>
<tr>
<td>HERMOSILLO .NEW-YORK</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>0.180</td>
</tr>
<tr>
<td>HERMOSILLO .CHICAGO</td>
<td>.</td>
<td>3.000</td>
<td>+INF</td>
<td></td>
</tr>
<tr>
<td>HERMOSILLO .DENVER</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>0.090</td>
</tr>
<tr>
<td>BILOXI .NEW-YORK</td>
<td>.</td>
<td>2.000</td>
<td>+INF</td>
<td></td>
</tr>
<tr>
<td>BILOXI .CHICAGO</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>0.180</td>
</tr>
<tr>
<td>BILOXI .DENVER</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>0.540</td>
</tr>
</tbody>
</table>
M4-5b add demand functions in each market.

Inverse demand functions in market j are given by

\[ P(J) = 4 - \frac{D(J)}{B(J)} \]

where \( P \) is price, \( D \) is demand, and \( B \) (parameter) is market size in J:

Doubling \( B \) means demand \( D \) doubles holding price constant.

Revenue and marginal revenue in market j are given by

\[ \text{revenue } j = P(J)D(J) = \left( 4 - \frac{D(J)}{B(J)} \right) D(J) \]

\[ MR(J) = 4 - 2\frac{D(J)}{B(J)} \]

\( MR \) not used in NLP version M4-5b)
inverse demand functions in market j are given by

\[ P(J) = 4 - D(J)/B(J) \]

where \( P \) is price, \( D \) is demand, and \( B \) (parameter) is market size in \( J \): doubling \( B \) means \( D \) doubles holding price constant.

Revenue and marginal revenue in market \( j \) are given by

\[ \text{revenue } j = P(J) \times D(J) = (4 - D(J)/B(J)) \times D(J) \]
\[ \text{MR}(J) = 4 - 2 \times D(J)/B(J) \]

SETS

I plants /GUANGDONG, HERMOSILLO, BILOXI/
J markets /NEW-YORK, CHICAGO, DENVER/;

PARAMETERS

A(I) plant capacity /GUANGDONG 4, HERMOSILLO 3, BILOXI 4/
B(J) market size j /NEW-YORK 12, CHICAGO 8, DENVER 4/
C(I) plant marginal cost of production /GUANGDONG 1, HERMOSILLO 1, BILOXI 3/
T(I,J) transport cost rate from market i to j
F freight rate parameter /90/;

TABLE DIST(I,J) distance

<table>
<thead>
<tr>
<th></th>
<th>NEW-YORK</th>
<th>CHICAGO</th>
<th>DENVER</th>
</tr>
</thead>
<tbody>
<tr>
<td>GUANGDONG</td>
<td>9</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>HERMOSILLO</td>
<td>4</td>
<td>2</td>
<td>1.5</td>
</tr>
<tr>
<td>BILOXI</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
DISPLAY DIST;

T(I,J) = F*DIST(I,J)/500;
DISPLAY C;

VARIABLES
PROF variable profit to be maximized;

NONNEGATIVE VARIABLES
X(I,J) shipment from i to j
D(J) demand in market j
LAMBDA shadow price on capacity constraint at plant I;

EQUATIONS
SUPPLY(I) supply constraint
DEMAND(J) demand constraint
PROFIT objective function;

SUPPLY(I).. A(I) =G= SUM(J, X(I,J));

DEMAND(J).. SUM(I, X(I,J)) =G= D(J);

PROFIT.. PROF =E= SUM(J, (4 - D(J)/B(J))*D(J)) - SUM((I,J), (C(I) + T(I,J))*X(I,J));

MODEL MNLP /PROFIT, SUPPLY, DEMAND/;

X.L(I,J) = 1;
D.L(J) = 1;

SOLVE MNLP USING NLP MAXIMIZING PROF;
B("CHICAGO") = 20;
SOLVE MNLP USING NLP MAXIMIZING PROF;
inverse demand functions in market $j$ are given by

\[ P(j) = 4 - \frac{D(j)}{B(j)} \]

where $P$ is price, $D$ is demand, and $B$ (parameter) is market size in $j$: doubling $B$ means $D$ doubles holding price constant. Revenue and marginal revenue in market $j$ are given by

\[ \text{revenue } j = P(j) \times D(j) = (4 - \frac{D(j)}{B(j)}) \times D(j) \]
\[ \text{MR}(j) = 4 - 2 \times \frac{D(j)}{B(j)} \]

SETS

I plants /GUANGDONG, HERMOSILLO, BILOXI/
J markets /NEW-YORK, CHICAGO, DENVER/;

PARAMETERS

A(I) plant capacity /GUANGDONG 4, HERMOSILLO 3, BILOXI 4/
B(J) market size j /NEW-YORK 12, CHICAGO 8, DENVER 4/
C(I) plant marginal cost of production /GUANGDONG 1, HERMOSILLO 1, BILOXI 3/
T(I,J) transport cost rate from market i to j
F freight rate parameter /90/
PROFIT profit - extracted after solve;

TABLE DIST(I,J) distance

<table>
<thead>
<tr>
<th></th>
<th>NEW-YORK</th>
<th>CHICAGO</th>
<th>DENVER</th>
</tr>
</thead>
<tbody>
<tr>
<td>GUANGDONG</td>
<td>9</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>HERMOSILLO</td>
<td>4</td>
<td>2</td>
<td>1.5</td>
</tr>
<tr>
<td>BILOXI</td>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
DISPLAY DIST;

T(I,J) = F*DIST(I,J)/500;
DISPLAY C;

NONNEGATIVE VARIABLES
X(I,J) shipment from i to j
D(J) demand in market j
LAMBDA(I) shadow price on capacity constraint at plant I;

EQUATIONS
SUPPLY(I) supply constraint
DEMAND(J) demand constraint
FOC(I,J) first order condition for X(I J) MC GE MR;

SUPPLY(I).. A(I) =G= SUM(J, X(I,J));

DEMAND(J).. SUM(I, X(I,J)) =G= D(J);

FOC(I,J).. C(I) + T(I,J) + LAMBDA(I) =G= 4 - 2*D(J)/B(J);

MODEL MMCP /SUPPLY.LAMBDA, DEMAND.D, FOC.X/;

X.L(I,J) = 1;
D.L(J) = 1;

SOLVE MMCP USING MCP;

PROFIT = SUM(J, (4 - D.L(J)/B(J))*D.L(J))
- SUM((I,J), (C(I) + T(I,J))*X.L(I,J));

DISPLAY PROFIT;
B("CHICAGO") = 20;
SOLVE MMCP USING MCP;

PROFIT = \textbf{SUM}(J, \ (4 - D.L(J)/B(J)) \ast D.L(J))
- \textbf{SUM}((I,J), \ (C(I) + T(I,J)) \ast X.L(I,J));

\textbf{DISPLAY} \ \textbf{PROFIT};
---- VAR X shipment from i to j (NLP version)

<table>
<thead>
<tr>
<th></th>
<th>LOWER</th>
<th>LEVEL</th>
<th>UPPER</th>
<th>MARGINAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>GUANGDONG .NEW-YORK</td>
<td>.</td>
<td>1.720</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>GUANGDONG .CHICAGO</td>
<td>.</td>
<td>0.280</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>GUANGDONG .DENVER</td>
<td>.</td>
<td>2.000</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>HERMOSILLO .NEW-YORK</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>-0.180</td>
</tr>
<tr>
<td>HERMOSILLO .CHICAGO</td>
<td>.</td>
<td>3.000</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>HERMOSILLO .DENVER</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>-0.090</td>
</tr>
<tr>
<td>BILOXI .NEW-YORK</td>
<td>.</td>
<td>2.120</td>
<td>+INF</td>
<td>EPS</td>
</tr>
<tr>
<td>BILOXI .CHICAGO</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>-0.180</td>
</tr>
<tr>
<td>BILOXI .DENVER</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>-0.540</td>
</tr>
</tbody>
</table>

---- VAR D demand in market j

<table>
<thead>
<tr>
<th></th>
<th>LOWER</th>
<th>LEVEL</th>
<th>UPPER</th>
<th>MARGINAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>NEW-YORK</td>
<td>.</td>
<td>3.840</td>
<td>+INF</td>
<td>EPS</td>
</tr>
<tr>
<td>CHICAGO</td>
<td>.</td>
<td>3.280</td>
<td>+INF</td>
<td>EPS</td>
</tr>
<tr>
<td>DENVER</td>
<td>.</td>
<td>2.000</td>
<td>+INF</td>
<td>.</td>
</tr>
</tbody>
</table>
## Counterfactual: make Chicago bigger

(NLP version)

### VAR X shipment from i to j

<table>
<thead>
<tr>
<th>Source</th>
<th>Destination</th>
<th>Lower</th>
<th>Level</th>
<th>Upper</th>
<th>Marginal</th>
</tr>
</thead>
<tbody>
<tr>
<td>GUANGDONG</td>
<td>NEW-YORK</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>-0.180</td>
</tr>
<tr>
<td>GUANGDONG</td>
<td>CHICAGO</td>
<td>.</td>
<td>2.458</td>
<td>+INF</td>
<td>EPS</td>
</tr>
<tr>
<td>GUANGDONG</td>
<td>DENVER</td>
<td>.</td>
<td>1.542</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>HERMOSILLO</td>
<td>NEW-YORK</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>-0.360</td>
</tr>
<tr>
<td>HERMOSILLO</td>
<td>CHICAGO</td>
<td>.</td>
<td>3.000</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>HERMOSILLO</td>
<td>DENVER</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>-0.090</td>
</tr>
<tr>
<td>BILOXI</td>
<td>NEW-YORK</td>
<td>.</td>
<td>3.547</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>BILOXI</td>
<td>CHICAGO</td>
<td>.</td>
<td>0.453</td>
<td>+INF</td>
<td>EPS</td>
</tr>
<tr>
<td>BILOXI</td>
<td>DENVER</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>-0.360</td>
</tr>
</tbody>
</table>

### VAR D demand in market j

<table>
<thead>
<tr>
<th>Market</th>
<th>Lower</th>
<th>Level</th>
<th>Upper</th>
<th>Marginal</th>
</tr>
</thead>
<tbody>
<tr>
<td>NEW-YORK</td>
<td>.</td>
<td>3.547</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>CHICAGO</td>
<td>.</td>
<td>5.911</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>DENVER</td>
<td>.</td>
<td>1.542</td>
<td>+INF</td>
<td>.</td>
</tr>
</tbody>
</table>
### VAR X shipment from i to j

<table>
<thead>
<tr>
<th></th>
<th>LOWER</th>
<th>LEVEL</th>
<th>UPPER</th>
<th>MARGINAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>GUANGDONG .NEW-YORK</td>
<td>.</td>
<td>1.720</td>
<td>+INF</td>
<td></td>
</tr>
<tr>
<td>GUANGDONG .CHICAGO</td>
<td>.</td>
<td>0.280</td>
<td>+INF</td>
<td></td>
</tr>
<tr>
<td>GUANGDONG .DENVER</td>
<td>.</td>
<td>2.000</td>
<td>+INF</td>
<td></td>
</tr>
<tr>
<td>HERMOSILLO .NEW-YORK</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>0.180</td>
</tr>
<tr>
<td>HERMOSILLO .CHICAGO</td>
<td>.</td>
<td>3.000</td>
<td>+INF</td>
<td></td>
</tr>
<tr>
<td>HERMOSILLO .DENVER</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>0.090</td>
</tr>
<tr>
<td>BILOXI .NEW-YORK</td>
<td>.</td>
<td>2.120</td>
<td>+INF</td>
<td></td>
</tr>
<tr>
<td>BILOXI .CHICAGO</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>0.180</td>
</tr>
<tr>
<td>BILOXI .DENVER</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>0.540</td>
</tr>
</tbody>
</table>

### VAR D demand in market j

<table>
<thead>
<tr>
<th></th>
<th>LOWER</th>
<th>LEVEL</th>
<th>UPPER</th>
<th>MARGINAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>NEW-YORK</td>
<td>.</td>
<td>3.840</td>
<td>+INF</td>
<td></td>
</tr>
<tr>
<td>CHICAGO</td>
<td>.</td>
<td>3.280</td>
<td>+INF</td>
<td></td>
</tr>
<tr>
<td>DENVER</td>
<td>.</td>
<td>2.000</td>
<td>+INF</td>
<td></td>
</tr>
</tbody>
</table>

### VAR LAMBDA shadow price on capacity constraint at plant I

<table>
<thead>
<tr>
<th></th>
<th>LOWER</th>
<th>LEVEL</th>
<th>UPPER</th>
<th>MARGINAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>GUANGDONG</td>
<td>.</td>
<td>0.740</td>
<td>+INF</td>
<td></td>
</tr>
<tr>
<td>HERMOSILLO</td>
<td>.</td>
<td>1.820</td>
<td>+INF</td>
<td></td>
</tr>
<tr>
<td>BILOXI</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>1.880</td>
</tr>
</tbody>
</table>
---- VAR X shipment from i to j (Chicago bigger, MCP version)

<table>
<thead>
<tr>
<th></th>
<th>LOWER</th>
<th>LEVEL</th>
<th>UPPER</th>
<th>MARGINAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>GUANGDONG . NEW-YORK</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>0.180</td>
</tr>
<tr>
<td>GUANGDONG . CHICAGO</td>
<td>.</td>
<td>2.458</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>GUANGDONG . DENVER</td>
<td>.</td>
<td>1.542</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>HERMOSILLO . NEW-YORK</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>0.360</td>
</tr>
<tr>
<td>HERMOSILLO . CHICAGO</td>
<td>.</td>
<td>3.000</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>HERMOSILLO . DENVER</td>
<td>.</td>
<td>.</td>
<td>+INF</td>
<td>0.090</td>
</tr>
<tr>
<td>BILOXI    . NEW-YORK</td>
<td>.</td>
<td>3.547</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>BILOXI    . CHICAGO</td>
<td>.</td>
<td>0.453</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>BILOXI    . DENVER</td>
<td>.</td>
<td>0.453</td>
<td>+INF</td>
<td>0.360</td>
</tr>
</tbody>
</table>

---- VAR D demand in market j

<table>
<thead>
<tr>
<th></th>
<th>LOWER</th>
<th>LEVEL</th>
<th>UPPER</th>
<th>MARGINAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>NEW-YORK</td>
<td>.</td>
<td>3.547</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>CHICAGO</td>
<td>.</td>
<td>5.911</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>DENVER</td>
<td>.</td>
<td>1.542</td>
<td>+INF</td>
<td>.</td>
</tr>
</tbody>
</table>

---- VAR LAMBDA shadow price on capacity constraint at plant I

<table>
<thead>
<tr>
<th></th>
<th>LOWER</th>
<th>LEVEL</th>
<th>UPPER</th>
<th>MARGINAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>GUANGDONG</td>
<td>.</td>
<td>0.969</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>HERMOSILLO</td>
<td>.</td>
<td>2.049</td>
<td>+INF</td>
<td>.</td>
</tr>
<tr>
<td>BILOXI</td>
<td>.</td>
<td>0.049</td>
<td>+INF</td>
<td>.</td>
</tr>
</tbody>
</table>
Exercise 5  Refinery scheduling problem

A refinery has one input, crude oil (CO), and produces 3 outputs:
  Gasoline
  Diesel
  Kerosene

The technology is call a Constant Elasticity of Transformation (CET) function, producing multiple outputs from one input.

\[
\left( \sum_i \alpha_i \left( \frac{X_i}{\alpha_i} \right)^\beta \right)^{\frac{1}{\beta}} = CO \quad \infty \geq \beta \geq 1, \quad \sigma = \frac{1}{\beta - 1}
\]

Note that if \( \beta = 2 \), for example, this is just the equation of a circle. This special case has an elasticity of transformation \( \sigma = 1 \).

If there are only two outputs, we would simply call this the PPF.
Let $p_i$ denote the price of product $i$. The “unit revenue function” is a value function, the *maximum* revenue from one unit of input.

$$r(p) = \max_x \sum p_i x_i + \lambda \left[ \sum_{i} a_i \left( \frac{x_i}{a_i} \right)^\beta \right]^{\frac{1}{\beta}} - 1$$

If you do the algebra, this unit value function is given by

$$r(p) = \left( \sum a_i p_i^{\sigma+1} \right)^{\frac{1}{\sigma+1}} \quad R(p) = \left( \sum a_i p_i^{\sigma+1} \right)^{\frac{1}{\sigma+1}} CO$$

where $R(p)$ is the total revenue derived from CO units of input. Applying Shepard’s lemma to $R(p)$, optimal outputs are

$$\frac{\partial R(p)}{\partial p_i} = X_i = ????$$
Exercise 5

(A) given the revenue function, apply Shepard’s lemma to get the optimal supply functions for the three products

(B) solve for optimal product outputs using exercise-q5.gms
   NLP formulation
   MCP formulation