Slides for Chapter 2: Examples of economic equilibrium problems translated into GAMS

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Tools of Economic Analysis   -   introducing the "third way"

(1)   Traditional analytical theory models   ("pencil and paper" technology)

(2)   Econometric estimation and testing.

(3)   Computer simulation modeling - the focus of this course

      (A)   for use in economic theory: allows us to construct and analyze theories that are much more complex than those tractable in traditional analytical models

"Many branches of both pure and applied mathematics are in great need of computing instruments to break the present stalemate created by the failure of the purely analytical approach to nonlinear problems"

--- John Von Neumann, 1945

(B)   for use in calculating counter-factual outcomes (e.g., due to policy changes) in empirical models

(C)   in both cases, for calculating numerical values for welfare changes and optimal values for policy variables.

(4)   Two ways of formulating economic models

(A)   as an constrained optimization problem

(B)   as a complementarity problem: square system of equations/inequalities and unknowns

2.1   Simple supply-demand problem illustrating complementarity

Supply and demand model of a single market, a partial equilibrium model. Two equations, supply and demand, two variables, price and quantity.

Economic equilibrium problems are thus represented as a system of n equations/inequalities in n unknowns.

Complementarity involves
  (a) associating each equation with a particular variable, called the complementary variable.

  (b) if the variables are restricted to be non-negative (prices and quantities), then the equations are written as weak inequalities.

If the equation holds as an equality in equilibrium, then the complementary variable is generally strictly positive.

If the equation holds as a strict inequality in equilibrium, the complementary variable is zero.

Supply of good X with price P.   The supply curve exploits the firm's optimization decision, equating price with marginal cost: $P = MC.$

$MC \geq P$ with the complementarity condition that $X \geq 0$

Note that the *price* equation is complementary with a *quantity* variable.

Suppose that COST $= aX + (b/2)X^2$.

Marginal cost is then given by $MC = a + bX$.

   $a + bX \geq P$ complementary with $X \geq 0$.

Optimizing consumer utility for a given income and prices will yield a demand function of the form $X = D(P, M)$ where M is income.

   $X \geq D(P, M)$ with the complementary condition that $P \geq 0$.

Note that the *quantity* equation is complementary with a *price* variable.

We will suppress income and assume a simple function:

$X = c + dP$ where $c > 0$, $d < 0$.

$X \geq c + dP$ complementary with $P \geq 0$.

How do we know which inequality is associate with which variable and the direction of the inequality?

Economic theory tells you which variable must be associated with which inequality and which way the inequality goes.

(1) ask the question whether or not a particular direction of the inequality is consistent with economic equilibrium.

(2) ask the question, "what must be true if the inequality is strict in equilibrium"?

$MC < P$ cannot be an equilibrium, there is a profit opportunity and more of the good will be produced.

$MC > P$ can be an equilibrium: the good is unprofitable and is not produced. Thus X is complementary with the supply equation.

$X < D(P,M)$ cannot be an equilibrium, excess demand will cause the price to rise and more will be supplied.

$X > D(P,M)$ can be an equilibrium, this must mean that the good is free, and $P = 0$. Thus P is complementary with the demand equation.

2. Coding an economic equilibrium problem in GAMS

First, comment statements can be used at the beginning of the code, preceded with a *, in the *first column* of a line. Actual code is shown in courier font.

```
$TITLE: M2-1.GMS  introductory model using MCP
* simple supply and demand model (partial equilibrium)
```

Now we can begin a series of declaration and assignment statements.

```
PARAMETERS
 A       intercept of supply on the P axis (MC at Q = 0)
 B       slope of supply:  this is dP over dQ
 C       demand on the Q axis (demand at P = 0)
 D       (inverse) slope of demand, dQ over dP
 TAX     a tax rate used later for experiments;
```

Parameters must be assigned values before the model is solved
These parameter values are referred to as *Case 1* below.

```
A = 2;
C = 6;
B = 1;
D = -1;
TAX = 0;
```

Now we declare a list of variables. They must be restricted to be
positive to make any economic sense, so declaring them as "positive
variables" tells GAMS to set lower bounds of zero on these variables.

```
NONNEGATIVE VARIABLES
  P     price of good X
  X     quantity of good X;
```

Now we similarly declare a list of equations. We can name them

anything we want provided it is a name not otherwise in use or, of course, a keyword.

```
EQUATIONS
 DEMAND supply relationship (mc cost ge price)
 SUPPLY quantity demanded as a function of price;
```

Specify equations. Format: equation name followed by two periods (full stops).

Then after a space, the equation is written out, with =G= GAMS code for "greater than or equal to".

```
SUPPLY..     A + B*X   =G=   P;


DEMAND..     X   =G=   C + D*P;
```

Next we need to declare a model: a set of equations and unknowns. The format is the keyword *model,* followed by a model name of your choosing. Here we use *equil.*

Next comes a "/" followed by a list of the equation names: each equation ends with a period followed by the name of the complementary variable.

```
MODEL EQUIL /SUPPLY.X, DEMAND.P/;
```

Finally, we need to tell GAMS to solve the model and what software is needed (GAMS does many other types of problems, such as optimization)

```
SOLVE EQUIL USING MCP;
```

This example uses parameter values which generate an "interior solution", meaning that both X and P are strictly positive.

Case 2: the good, or a particular way to produce or obtain a good (e.g., via imports) is too expensive relative to some alternative: production or trade activity is not used in equilibrium. This activity is "slack" in equilibrium.
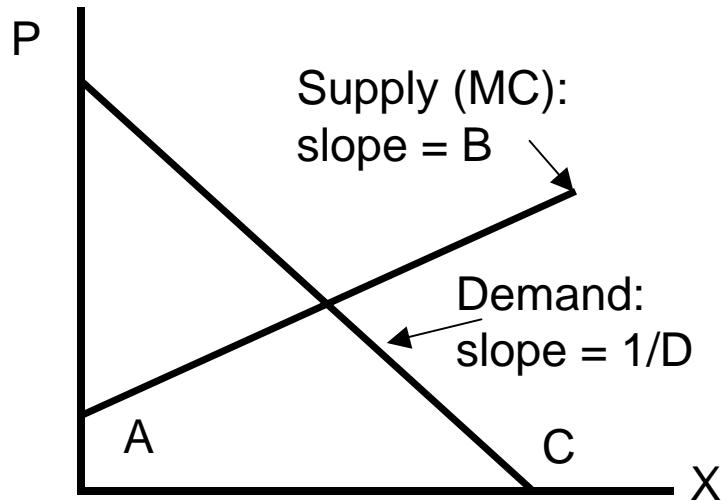
```
A = 7;
SOLVE EQUIL USING MCP;
```

Case 3: The final possibility is that a good or factor of production may be so plentiful that it commands a zero price in equilibrium
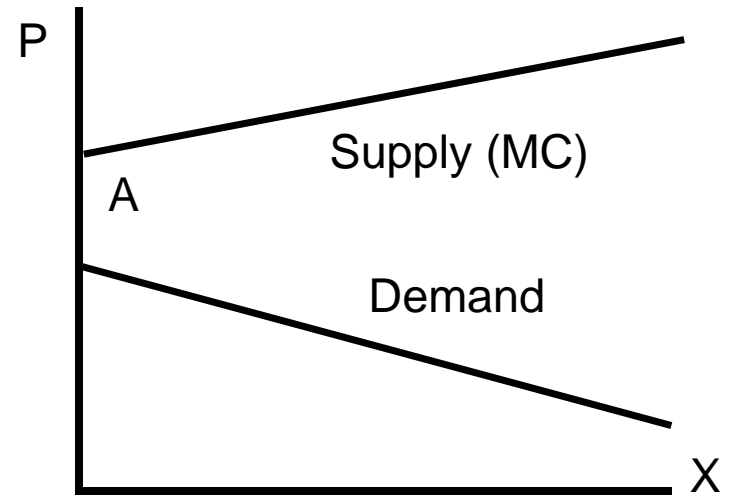
```
A = -7;
SOLVE EQUIL USING MCP;
```
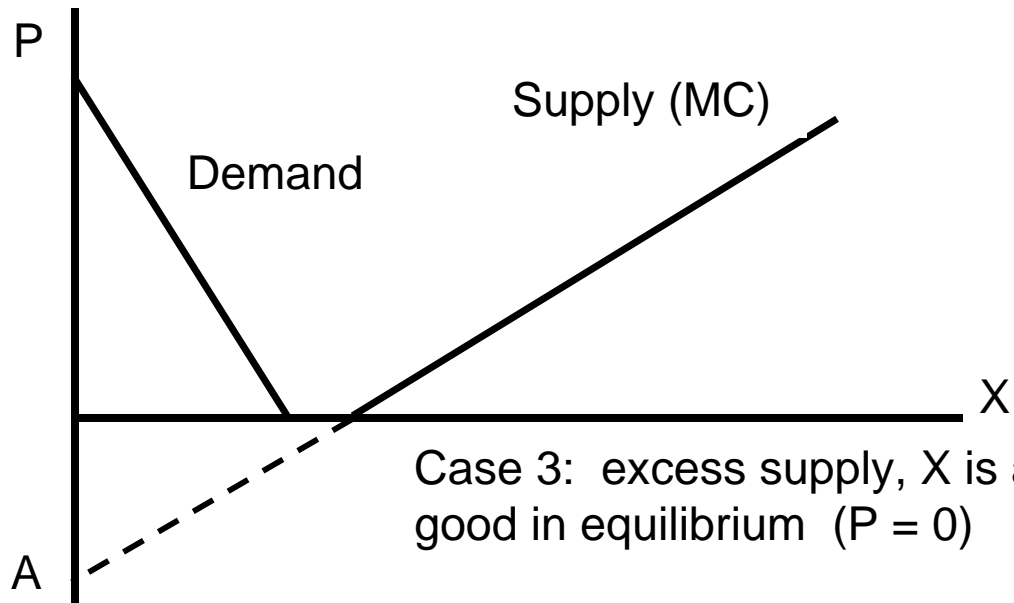
Figure 1: Three outcomes of partial equilibrium example

Supply (MC): slope = B

Demand: slope = 1/D

A

C

Case 1: interior solution

Supply (MC)

A

Demand

Case 2: X is too expensive, not produce in equilibrium (X = 0)

Demand

Supply (MC)

A

Case 3: excess supply, X is a free good in equilibrium (P = 0)

3. Counterfactual (comparative statics) experiments

In the following example, we use a parameter called "TAX", an ad valorem tax on the input (marginal cost) to production.

To do so, we declare and specify an additional equation "SUPPLY2" and model "EQUIL2" (GAMS does not allow us to simply rewrite the equation or use the old model name once we name a new equation).

Three parameters are declared and used to extract information after we solve the model.

```
PARAMETERS
 CONSPRICE     consumer price
 PRODPRICE     producer price (equal to marginal cost)
 TAXREV        tax revenue (tax base is producer price);
```

```
EQUATIONS
  SUPPLY2;

SUPPLY2..      (A + B*X)*(1+TAX)   =G=   P;

MODEL EQUIL2 /SUPPLY2.X, DEMAND.P/;
```

Before we solve the model, we need to first return the supply
intercept to its original value: parameter changes are kept as
permanent unless "undone" in the code.

Then we resolve the base case (strictly speaking, this is not
necessary).

```
A = 2;
TAX = 0;
SOLVE EQUIL2 USING MCP;
```

Now set the tax at 25% and solve the model.

```
TAX = 0.25;
SOLVE EQUIL2 USING MCP;
```

After we solve the model, it is common to extract output using GAMS
    code.

Note that with the tax specified on inputs, the price being solved for is
    the *consumer price* (price paid by the consumer) not the *producer
    price* (price received by the producer).

Producer price is the same as marginal cost.

GAMS stores three values for variables: its current value, its upper
    bound (infinity in our case) and its lower bound (zero in our case).

When the modeler wants the value of a variable, you have to use the notation *NAME.L* where the *L* stands for level.

GAMS does not automatically print out the values of parameters in the solution, so you have to ask GAMS to *DISPLAY* these values.

```
CONSPRICE = P.L;
PRODPRICE = P.L/(1+TAX);
TAXREV = PRODPRICE*TAX*X.L;

DISPLAY CONSPRICE, PRODPRICE, TAXREV;
```

4. Reading the output

Again, you need to consult the GAMS users' manual to see how to run the model, and to read and interpret output once you run the model.

There is a lot of "stuff" in GAMS listing (output) files, which are stored as *FILENAME.LST* after you run the model (*LST* is for "listing file"). Here are just the relevant parts of our model runs

```
              S O L V E           S U M M A R Y


      MODEL     EQUIL
      TYPE      MCP
      SOLVER    MILES                       FROM LINE   50


****  SOLVER STATUS       1 NORMAL COMPLETION
****  MODEL STATUS        1 OPTIMAL
```

Case 1: The *LEVEL* is the solution value of the variables. *MARGINAL* indicates the degree to which the equation corresponding to the variable is out of equality.

For *P* (price), the equation is *DEMAND* and the value of the marginal is supply minus demand. For *X* (quantity), the equation is *SUPPLY* and the value of the marginal is the excess of marginal cost over price.

Variables that have *positive values* in the solution should have *zero marginals*.

Variables that have *zero values* in the solution should have *positive marginals*.

```
                   LOWER        LEVEL        UPPER       MARGINAL


---- VAR P           .          4.000        +INF
---- VAR X           .          2.000        +INF
```

Case 2: This is the zero-output case. The price equation holds, but the quantity equation is slack. The marginal of 1.0 indicates that, at the solution, marginal cost exceed price by 1.0.

	LOWER	LEVEL	UPPER	MARGINAL
---- VAR P	.	6.000	+INF	.
---- VAR X	.	.	+INF	1.000

Case 3: This is the free-good case. Now the price equation is slack, and the marginal of 1.0 indicates that, at the solution, supply exceeds demand by 1.0.

	LOWER	LEVEL	UPPER	MARGINAL
---- VAR P	.	.	+INF	1.000
---- VAR X	.	7.000	+INF	.

Tax example: Here is the tax example (after the supply intercept is set back at 2.0). Note the "tax incidence" is split between the producer and consumer: the initial price was 4.0.

```
                   LOWER        LEVEL        UPPER       MARGINAL

---- VAR P           .          4.444        +INF           .
---- VAR X           .          1.556        +INF           .

----   82 PARAMETER CONSPRICE = 4.444   consumer price

----   82 PARAMETER PRODPRICE = 3.556   producer price

----   82 PARAMETER TAXREV        = 1.383   tax revenue
```

```
$TITLE: M2-1.GMS  introductory model using MCP
* simple supply and demand model (partial equilibrium)
```

PARAMETERS
```
 A          intercept of supply on the P axis (MC at Q = 0)
 B          change in MC in response to Q - this is dP over dQ
 C          intercept of demand on the Q axis (demand at P = 0)
 D          response of demand to changes in price - dQ over dP
 TAX        a tax rate used later for experiments;


A = 2;
C = 6;
B = 1;
D = -1;
```

NONNEGATIVE VARIABLES
```
 P          price of good X
 X          quantity of good X;
```

EQUATIONS
```
 SUPPLY     supply relationship (marginal cost ge price)
 DEMAND     quantity demanded as a function of price;

SUPPLY..    A + B*X  =G=  P;
```

```
DEMAND..     X  =G=  C + D*P;



MODEL EQUIL /SUPPLY.X, DEMAND.P/;
OPTION MCP = PATH;
SOLVE EQUIL USING MCP;


* counter factual 1: shift the supply curve (marginal cost) up/left


A = 7;
SOLVE EQUIL USING MCP;


* country factual 2: shift the supply curve (marginal cost) down/right


A = -7;
SOLVE EQUIL USING MCP;




* exercise 1: extract economic information from the solution


PARAMETERS
 CONSPRICE    consumer price
 PRODPRICE    producer price (equal to marginal cost)
 TAXREV       tax revenue (note tax base is producer price)
```

```
  CONSURP1      consumer surplus with no tax
  CONSURP2      consumer surplus with 25% tax
  PROSURP1      producer "surplus" with no tax
  PROSURP2      producer "surplus" with 25% tax
  DWL           net loss from the tax;


EQUATIONS
 SUPPLY2;

SUPPLY2..      (A + B*X)*(1+TAX)  =G=  P;

MODEL EQUIL2 /SUPPLY2.X, DEMAND.P/;

A = 2;
TAX = 0;
SOLVE EQUIL2 USING MCP;

CONSURP1 = (-C/D - P.L)*X.L/2;
PROSURP1 = (P.L/(1+TAX) - A)*X.L/2;

TAX = 0.25;
SOLVE EQUIL2 USING MCP;

CONSURP2 = (-C/D - P.L)*X.L/2;
PROSURP2 = (P.L/(1+TAX) - A)*X.L/2;
```

```
CONSPRICE = P.L;
PRODPRICE = P.L/(1+TAX);
TAXREV = PRODPRICE*TAX*X.L;
DISPLAY CONSPRICE, PRODPRICE, TAXREV;


DWL = CONSURP1 + PROSURP1 - (CONSURP2 + PROSURP2 + TAXREV);
DISPLAY CONSURP1, PROSURP1, CONSURP2, PROSURP2, TAXREV, DWL;



*exercise 2, mismatch the complementary variables

TAX = 0;

MODEL EQUIL3 /SUPPLY.P, DEMAND.X/;

SOLVE EQUIL3 USING MCP;

P.L = 0;
X.L = 6;

A = 7;
SOLVE EQUIL3 USING MCP;

A = -7;
SOLVE EQUIL3 USING MCP;
```

2.2 Maximization of utility subject to a linear budget constraint

Illustrates the use of the GAMS NLP and MCP solvers

NLP non-linear programming

MCP mixed complementarity problem

Cobb-Douglas utility function with linear budget constraint

$$U \; = \; \left(\frac{X_1}{\alpha}\right)^{\alpha}\left(\frac{X_2}{1-\alpha}\right)^{1-\alpha} \qquad M \; = \; p_1 X_1 \; + \; p_2 X_2$$

Result: C-D exponents are expenditure shares: $\quad \alpha \; = \; \dfrac{p_1 X_1}{M}$

"Primal" formulation as an optimization problem

$$\max \quad U = X_1^{\alpha} X_2^{1-\alpha} + \lambda(M - p_1 X_1 - p_2 X_2)$$

$$\frac{\partial U}{\partial X_1} = \alpha X_1^{\alpha-1} X_2^{1-\alpha} - \lambda p_1 \leq 0$$

$$\frac{\partial U}{\partial X_2} = (1-\alpha) X_1^{\alpha} X_2^{-\alpha} - \lambda p_2 \leq 0$$

$$\frac{\partial U}{\partial \lambda} = M - p_1 X_1 - p_2 X_2 \leq 0$$

Can solve for "Marshallian" or "uncompensated" demand functions

$$X_i = D_i(p_1, p_2, M)$$

"Dual" formulation as a minimization problem

$$\min \ p_1 X_1 + p_2 X_2 + \mu(U - X_1^{\alpha} X_2^{1-\alpha})$$

$$\frac{\partial E}{\partial X_1} = p_1 - \mu \alpha X_1^{\alpha-1} X_2^{1-\alpha} \leq 0$$

$$\frac{\partial E}{\partial X_2} = p_2 - \mu(1-\alpha) X_1^{\alpha} X_2^{-\alpha} \leq 0$$

$$\frac{\partial E}{\partial \mu} = U - X_1^{\alpha} X_2^{1-\alpha} \leq 0$$

Can solve for "Hicksian" or "compensated" demand functions

$$X_i = H_i(p_1, p_2, U)$$

```
$TITLE: M2-2.GMS: consumer choice, modeled as an NLP and a MCP
*     maximize utility subject to a linear budget constraint
*     two goods, Cobb-Douglas preferences

$ONTEXT
This program introduces economic students to GAMS and GAMS solvers.
The problem itself is known and loved by all econ students from
  undergraduate intermediate micro economics on up:
Maximizing utility with two goods and a linear budget constraint.

Four versions are considered
 OPTIMIZE: direct constrained optimization using the NLP
           (non-linear programming) solver
 COMPLEM:  uses the first-order conditions (FOC) to create a square
           system of n inequalities in n unknowns, solved using the MCP
           (mixed complementarity problem) solver
 COMPLEM2: instead of the utility function and FOC, uses the expenditure
           function and Marshallian demand functions, solved as an MCP
 COMPLEM3: instead of the utility function and FOC, uses the expenditure
           function and Hicksian demand functions, solved as an MCP
$OFFTEXT

PARAMETERS
 M                 Income
 P1, P2            prices of goods X1 and X2
 S1, S2            utility shares of X1 and X2;
```

```
M = 100;
P1 = 1;
P2 = 1;
S1 = 0.5;
S2 = 0.5;
```

NONNEGATIVE VARIABLES

```
   X1, X2    Commodity demands
   LAMBDA    Lagrangean multiplier (marginal utility of income);
```

VARIABLES

```
   U         Welfare;
```

EQUATIONS

```
   UTILITY      Utility
   INCOME       Income-expenditure constraint
   FOC1, FOC2   First-order conditions for X1 and X2;

UTILITY..    U =E= 2*(X1**S1)*(X2**S2);

INCOME..     M =G= P1*X1 + P2*X2;
```

```
FOC1..          LAMBDA*P1 =G= 2*S1*X1**(S1-1)*(X2**S2);


FOC2..          LAMBDA*P2 =G= 2*S2*X2**(S2-1)*(X1**S1);



* set starting values
U.L = 100;
X1.L = 50;
X2.L = 50;
LAMBDA.L = 1;

* modeled as a non-linear programming problem


MODEL OPTIMIZE /UTILITY, INCOME/;
SOLVE OPTIMIZE USING NLP MAXIMIZING U;

* modeled as a complementarity problem


MODEL COMPLEM /UTILITY.U, INCOME.LAMBDA, FOC1.X1, FOC2.X2/;
SOLVE COMPLEM USING MCP;

* counterfactuals


P1 = 2;


SOLVE OPTIMIZE USING NLP MAXIMZING U;
```

```
SOLVE COMPLEM USING MCP;


P1 = 1;
M = 200;


SOLVE OPTIMIZE USING NLP MAXIMZING U;
SOLVE COMPLEM USING MCP;



* now use the expenditure function, giving the minimum cost of buying
* one unit of utility: COSTU = P1**S1 * P2**S2 = PU
* where PU is the "price" of utility: the inverse of lambda
* two versions are presented:
* one using Marshallian (uncompensated) demand: X_i = F_i(P1, P2, M)
* one using Hicksian (compensated) demand: X_i = F_i(P1, P2, U)


P1 = 1;
M = 100;


NONNEGATIVE VARIABLES
    PU          price of utility;


EQUATIONS
    COSTU       expenditure function: cost of producing utility = PU
    DEMANDM1    Marshallian demand for good 1
    DEMANDM2    Marshallian demand for good 2
```

```
    DEMANDH1   Hicksian demand for good 1
    DEMANDH2   Hicksian demand for good 2
    DEMANDU    Demand for utility (indirect utility function);


COSTU..     P1**S1 * P2**S2 =G= PU;


DEMANDM1.. X1 =G= S1*M/P1;


DEMANDM2.. X2 =G= S2*M/P2;


DEMANDH1.. X1 =G= S1*PU*U/P1;


DEMANDH2.. X2 =G= S2*PU*U/P2;


DEMANDU..  U =E= M/PU;


PU.L = 1;


MODEL COMPLEM2 marshall /COSTU.U, DEMANDM1.X1, DEMANDM2.X2, DEMANDU.PU/;
MODEL COMPLEM3 hicks    /COSTU.U, DEMANDH1.X1, DEMANDH2.X2, DEMANDU.PU/;


SOLVE COMPLEM2 USING MCP;
SOLVE COMPLEM3 USING MCP;


* counterfactuals
```

```
P1 = 2;

SOLVE COMPLEM2 USING MCP;
SOLVE COMPLEM3 USING MCP;


P1 = 1;
M = 200;


SOLVE COMPLEM2 USING MCP;
SOLVE COMPLEM3 USING MCP;
```

2.3   Extension of utility optimization: add a rationing constraint

Illustrate slackness in equilibrium, illustrate "shadow" prices
$RATION \geq X_1$ with Lagrangean multiplier λ_r

$$\max \quad U = X_1^\alpha X_2^{1-\alpha} + \lambda(M - p_1 X_1 - p_2 X_2) + \lambda_r(RATION - X_1)$$

$$\frac{\partial U}{\partial X_1} = \alpha X_1^{\alpha-1} X_2^{1-\alpha} - \lambda p_1 - \lambda_r \leq 0$$

$$\frac{\partial U}{\partial X_2} = (1-\alpha) X_1^\alpha X_2^{-\alpha} - \lambda p_2 \leq 0$$

$$\frac{\partial U}{\partial \lambda} = M - p_1 X_1 - p_2 X_2 \leq 0 \qquad \frac{\partial U}{\partial \lambda_r} = RATION - X_1 \leq 0$$

$TITLE: M2-3.GMS add a rationing constraint to model M2-2
*   MAXIMIZE UTILITY SUBJECT TO A LINEAR BUDGET CONSTRAINT
*   PLUS RATIONING CONSTRAINT ON X1
*   two goods, Cobb-Douglas preferences

PARAMETERS
 M   Income
 P1, P2   prices of goods X1 and X2
 S1, S2   util shares of X1 and X2
 RATION   rationing constraint on the quantity of X1;

M = 100;
P1 = 1;
P2 = 1;
S1 = 0.5;
S2 = 0.5;
RATION = 100.;

NONNEGATIVE VARIABLES
    X1, X2   Commodity demands
    LAMBDAI   Lagrangean multiplier (marginal utility of income)
    LAMBDAR   Lagrangean mulitplier on rationing constraint;

VARIABLES
    U   Welfare;

```
EQUATIONS
   UTILITY      Utility
   INCOME        Income-expenditure constraint
   RATION1       Rationing contraint on good X1
   FOC1, FOC2    First-order conditions for X1 and X2;


UTILITY..    U =E= 2*(X1**S1)*(X2**S2);


INCOME..     M =G= P1*X1 + P2*X2;


RATION1..    RATION =G= X1;


FOC1..       LAMBDAI*P1 + LAMBDAR =G= 2*S1*X1**(S1-1)*(X2**S2);


FOC2..       LAMBDAI*P2 =G= 2*S2*X2**(S2-1)*(X1**S1);


* modeled as a non-linear programming problem
* set starting values


U.L = 100;
X1.L = 50;
X2.L = 50;
LAMBDAI.L = 1;
LAMBDAR.L = 0;


MODEL OPTIMIZE /UTILITY, INCOME, RATION1/;
```

```
SOLVE OPTIMIZE USING NLP MAXIMIZING U;

* modeled as a complementarity problem

MODEL COMPLEM /UTILITY.U, INCOME.LAMBDAI, RATION1.LAMBDAR,
              FOC1.X1, FOC2.X2/;
SOLVE COMPLEM USING MCP;


* try binding rationing constraint at X1 <= RATION = 25;

RATION = 25;
SOLVE OPTIMIZE USING NLP MAXIMIZING U;
SOLVE COMPLEM USING MCP;

* show that shadow price of rationing constraint increases with income
* could lead to a black market in rationing coupons, "scalping" tickets

M = 200;
SOLVE OPTIMIZE USING NLP MAXIMIZING U;
SOLVE COMPLEM USING MCP;


* illustrate the mpec solver
* suppose we want to enforce the rationing contraint via licenses for X1
* consumers are given an allocation of licenses which is RATION
```

```
* PLIC is an endogenous variables whose value is the license price
* the value of the rationing license allocation should be treated as
* part of income



NONNEGATIVE VARIABLES
  PLIC;


EQUATIONS
  INCOMEa
  FOC1a;


M = 100;
RATION = 25;
U.L = 100;
X1.L = 25;
X2.L = 75;
PLIC.L = 0.1;


INCOMEa..    M + (PLIC*RATION) =E= (P1 + PLIC)*X1 + P2*X2 ;


FOC1a..      LAMBDAI*(P1 + PLIC) =G= 2*S1*X1**(S1-1)*(X2**S2);


MODEL MPEC      /UTILITY, INCOMEa.LAMBDAI, FOC1a.X1, FOC2.X2,
                   RATION1.PLIC/;
MODEL COMPLEM2 /UTILITY.U, INCOMEa.LAMBDAI, FOC1a.X1, FOC2.X2,
```

```
                    RATION1.PLIC/;


OPTION MPEC = nlpec;


SOLVE  MPEC USING MPEC MAXIMIZING U;
SOLVE  COMPLEM2 USING MCP;



M = 200;


SOLVE  MPEC USING MPEC MAXIMIZING U;
SOLVE  COMPLEM2 USING MCP;




* now use the expenditure function, giving the minimum cost of buying
* one unit of utility: COSTU = P1**S1 * P2**S2 = PU
* where PU is the "price" of utility: the inverse of lambda
* two versions are presented:
* one using Marshallian (uncompensated) demand: Xi = F(P1, P2, M)
* one using Hicksian (compensated) demand: Xi = F(P1, P2, U)


RATION = 100;
M = 100;


NONNEGATIVE VARIABLES
    PU          price of utility
    M1          income inclusive of the value of rationing allocation;
```

EQUATIONS
```
   COSTU      expenditure function: cost of producing utility = PU
   DEMANDM1   Marshallian demand for good 1
   DEMANDM2   Marshallian demand for good 2
   DEMANDH1   Hicksian demand for good 1
   DEMANDH2   Hicksian demand for good 2
   DEMANDU    Demand for utility (indirect utility function)
   RATION1b   Rationing constraint (same as before)
   INCOMEb    Income balance equation;
```

```
COSTU..    (PLIC+P1)**S1 * P2**S2 =G= PU;


DEMANDM1.. X1 =G= S1*M1/(P1+PLIC);


DEMANDM2.. X2 =G= S2*M1/P2;


DEMANDH1.. X1 =G= S1*PU*U/(P1+PLIC);


DEMANDH2.. X2 =G= S2*PU*U/P2;


DEMANDU..  U =E= M1/PU;


RATION1b.. RATION =G= X1;


INCOMEb..  M1 =E= M + PLIC*RATION;
```

```
PU.L = 1;

MODEL COMPLEM3 /COSTU.U, DEMANDM1.X1, DEMANDM2.X2, DEMANDU.PU,
                RATION1b.PLIC, INCOMEb.M1/;
MODEL COMPLEM4 /COSTU.U, DEMANDH1.X1, DEMANDH2.X2, DEMANDU.PU,
                RATION1b.PLIC, INCOMEb.M1/;


SOLVE COMPLEM3 USING MCP;
SOLVE COMPLEM4 USING MCP;

* counterfactuals


RATION = 25;

SOLVE COMPLEM3 USING MCP;
SOLVE COMPLEM4 USING MCP;


M = 200;

SOLVE COMPLEM3 USING MCP;
SOLVE COMPLEM4 USING MCP;



*$exit
```

```
* scenario generation

SETS I indexes different values of rationing constraint /I1*I10/
     J indexes income levels /J1*J10/;

PARAMETERS
 RLEVEL(I)
 PCINCOME(J)
 LICENSEP(I,J);

U.L = 50;
X1.L = 25;
X2.L = 25;
PLIC.L = 0.;
LAMBDAI.L = 1;

* the following is to prevent solver failure when evaluating X1**(S1-1)
* at X1 = 0 (given S1-1 < 0)
X1.LO = 0.01;
X2.LO = 0.01;

LOOP(I,
LOOP(J,

 RATION = 110 - 10*ORD(I);
 M = 25 + 25*ORD(J);
```

```
SOLVE MPEC USING MPEC MAXIMIZING U;

 RLEVEL(I) = RATION;
 PCINCOME(J) = M;
 LICENSEP(I,J) = PLIC.L;


);
);


DISPLAY RLEVEL, PCINCOME, LICENSEP;


$LIBINCLUDE XLDUMP LICENSEP M2-3.XLS SHEET1!B3
```

```
$TITLE: M2-4.GMS quick introduction to sets and scenarios using M2-2
*    MAXIMIZE UTILITY SUBJECT TO A LINEAR BUDGET CONSTRAINT
*    same as UTIL-OPT1.GMS but introduces set notation


SET I  Prices and Goods / X1, X2  /;
ALIAS (I, II);


PARAMETER
 M              Income
 RATION         ration of X1 (constraint on max consumption of X1)
 P(I)           prices
 S(I)           util shares;


M = 100;
P("X1") = 1;
P("X2") = 1;
S("X1") = 0.5;
S("X2") = 0.5;
RATION  = 100;



NONNEGATIVE VARIABLES

   X(I)       Commodity demands
   LAMBDAI    Marginal utility of income (Lagrangean multiplier)
   LAMBDAR    Marginal effect of ration constraint;
```

```
VARIABLES

    U           Welfare;

EQUATIONS

    UTILITY
    INCOME
    RATION1
    FOC(I);

UTILITY..    U =E= 2*PROD(I, X(I)**S(I));

INCOME..     M =G= SUM(I, P(I)*X(I));

RATION1..    RATION =G= X("X1");

FOC(I)..     LAMBDAI*P(I) + LAMBDAR$(ORD(I) EQ 1) =G=
                S(I)*X(I)**(-1)*2*PROD(II, X(II)**S(II));

U.L = 100;
X.L(I) = 50;
RATION = 100;
```

```
*  first, solve the model as an nlp, max U subject to income
*  rationing constraint in non-binding

MODEL UMAX /UTILITY, INCOME, RATION1/;
SOLVE UMAX USING NLP MAXIMIZING U;

*  second, solve the model as an mcp, using the two FOC and income
LAMBDAI.L = 1;
LAMBDAR.L = 0;

MODEL COMPLEM /UTILITY.U, INCOME.LAMBDAI, RATION1.LAMBDAR, FOC.X/;
SOLVE COMPLEM USING MCP;



* scenario generation
SETS J indexes different values of rationing constraint /J1*J10/;

PARAMETERS
 RLEVEL(J)
 WELFARE(J)
 LAMRATION(J)
 RESULTS(J, *);

LOOP(J,
 RATION = 110 - 10*ORD(J);
```

```
SOLVE COMPLEM USING MCP;

 RLEVEL(J) = RATION;
 WELFARE(J) = U.L;
 LAMRATION(J) = LAMBDAR.L;

);


RESULTS(J, "RLEVEL") = RLEVEL(J);
RESULTS(J, "WELFARE") = WELFARE(J);
RESULTS(J, "LAMRATION") = LAMRATION(J);

DISPLAY RLEVEL, WELFARE, LAMRATION, RESULTS;

$LIBINCLUDE XLDUMP RESULTS M2-3.XLS SHEET2!B3
```

2.5   Toward General Equilibrium

We used a partial-equilibrium model to introduce complementarity and the GAMS solver. Now we want to take a step toward general equilibrium.

One household and one firm. The household has a fixed stock of labor (L).

Further assumption: the household derives no utility from leisure, and so will always supply all labor to the market if the wage rate (W) is positive.

There is a single good (X) produced from the single input labor.

The firm buys labor services and sells X to the household at price P.

The household receives income from selling labor services and uses it all to buy X.

Our model is shown schematically in Figure 2.   As noted in the Figure, general equilibrium requires five conditions:

    (1) consumer optimization
    (2) producer optimization
    (3) supply $\geq$ demand for labor
    (4) supply $\geq$ demand for X
    (5) consumer income equals expenditure (income-expenditure balance)

The consumer has no alternative use for labor and so optimizes by supplying it all.

The consumer prefers more X and so the demand for X is going to be given by $X = WL/P$: it is optimal to spend all income on X.

Figure 2: General equilibrium
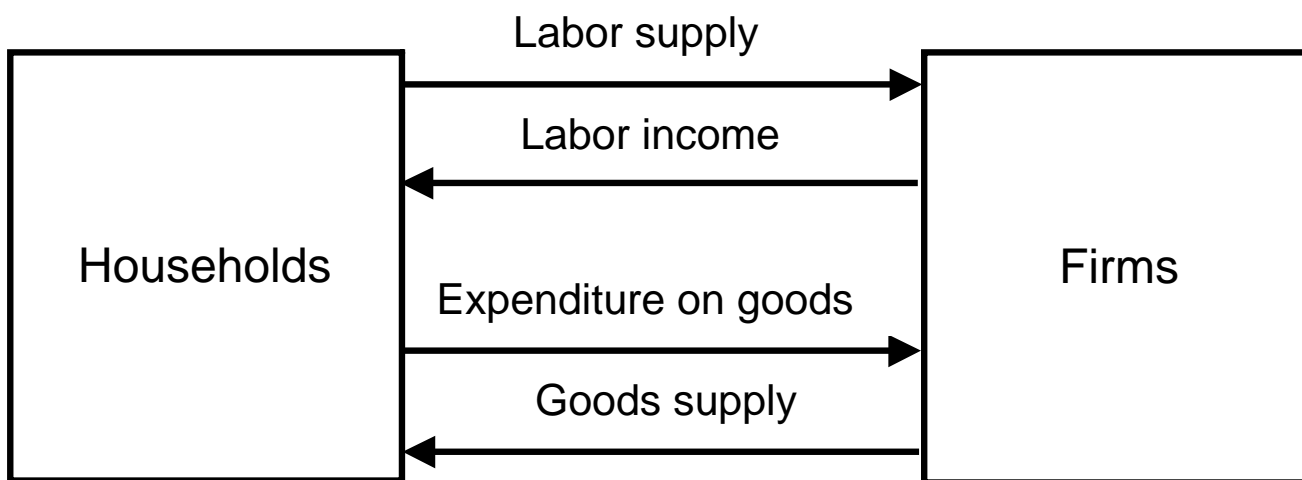
Conditions for general equilibrium

(1)   Firms optimize

(2)   Consumers optimize

(3)   Supply equals demand for labor

(4)   Supply equals demand for good X

(5)   Income equals expenditure (all labor income is spent on X)

2.5   Toward General Equilibrium

We used a partial-equilibrium model to introduce complementarity and the GAMS solver. Now we want to take a step toward general equilibrium.

One household and one firm. The household has a fixed stock of labor (L).

Further assumption: the household derives no utility from leisure, and so will always supply all labor to the market if the wage rate (W) is positive.

There is a single good (X) produced from the single input labor.

The firm buys labor services and sells X to the household at price P.

The household receives income from selling labor services and uses it all to buy *X*.

Our model is shown schematically in Figure 2.   As noted in the Figure, general equilibrium requires five conditions:

(1) consumer optimization
(2) producer optimization
(3) supply $\geq$ demand for labor
(4) supply $\geq$ demand for *X*
(5) consumer income equals expenditure (income-expenditure balance)

The consumer has no alternative use for labor and so optimizes by supplying it all.

The consumer prefers more X and so the demand for *X* is going to be given by $X = WL/P$: it is optimal to spend all income on *X*.

The production function for *X* will *be given as X = αL, where α is the marginal product of labor in* X production.

We assume competition and free entry, so that the firm is force to price at marginal cost.

Marginal cost, the cost of producing one more unit of *X*, is given by W/α (1/α is the amount of labor needed for one unit of *X*).

There are two parameters, total labor supply (*LBAR*) and productivity α (*ALPHA*).

```
PARAMETERS
  LBAR     labor supply (fixed and inelastic)
  ALPHA    productivity parameter  X = ALPHA*L;

LBAR = 100;
```

```
ALPHA = 2;
```

There will be four positive variables: price (P), quantity of X (X), the wage rate (W), and consumer income (INCOME).

```
POSITIVE VARIABLES
  P           price of X
  X           quantity of X
  W           wage rate
  INCOME      income from labor supply;
```

There are four equations. One important feature of general-equilibrium models with constant returns to scale, is that there is really no "supply" function in the usual partial-equilibrium sense:

At given factor prices, the "supply" curve (the marginal cost curve) is horizontal.

Supply is perfectly elastic at a fixed price $P = MC$, the latter being a constant at constant factor prices.

With constant returns, marginal cost is also average cost, and so this equation can also be thought of as a free-entry, zero-profits condition.

Accordingly, we refer to this equation as *ZPROFIT* (zero profits).

Then we require two market-clearing conditions.

First, supply of *X* must be greater than or equal to demand with the complementary variable being *P*, the price of *X*. This is referred to as *CMKTCLEAR* (commodity-market clearing).

Second, the supply of labor must be greater than or equal to its demand, with the complementary variable being W, the wage rate (price of labor). This is labeled *LMKTCLEAR* (labor-market clearing).

Finally, we require income balance: consumer expenditure equals labor income. This is labeled *CONSINCOME*.

```
EQUATIONS
  ZPROFIT        zeroprofits in X production
  CMKTCLEAR      commodity (X) market clearing
  LMKTCLEAR      labor market clearing
  CONSINCOME     consumer income;
```

Marginal cost is just the wage rate divided by α.

Commodity market clearing: *X* is greater than or equal to demand, which is just income divided by *P*, the price of *X*.

Labor-market clearing: supply, *LBAR*, is greater than or equal to labor demand, given by *X* divided by α (*X*/α).

Income spent on *X* must equal wage income, *W* times *LBAR* (*W*LBAR*).

```
ZPROFIT..      W/ALPHA =G= P;


CMKTCLEAR..   X =G= INCOME/P;


LMKTCLEAR..   LBAR =G= X/ALPHA;


CONSINCOME.. INCOME =G= W*LBAR;
```

We name the model "GE" for general equilibrium, and associate each equation in the model with its complementary variable.

```
MODEL GE /ZPROFIT.X, CMKTCLEAR.P, LMKTCLEAR.W,
                    CONSINCOME.INCOME/;
```

Setting starting values of variables. Helps the solver find the solution, important in complex problems. The notation for setting an initial value of a variable is the *NAME.L* notation we used earlier.

```
* set some starting values

P.L = 1;
W.L = 1;
X.L = 200;
INCOME.L = 100;
```

One final issue that is familiar to students of economics, is that there is an indeterminacy of the "price level" in this type of problem.

If $P = W = 1$ and $INCOME = 100$ are solutions to this model, so are any proportional multiples of these values such as $P = W = 2$, $INCOME = 200$.

More formally, there are really only three independent equations in this model, the fourth is automatically satified if three hold. This is known in economics as "Walras' Law".

The solution to this problem is to fix one price, termed the "numeraire".

Then all prices are measured relative or in terms of this numeraire. Suppose that we choose the wage rate to be the numeraire.

The notation is *W.FX* where *.FX* stands for "fix".

It is important to understand that *W.FX* is not the same as *W.L*.

The former holds W fixed throughout the remainder of the program, whereas *W.L* is just setting an initial value that will be changed by the solver.

When a variable is fixed, GAMS automatically drops the complementary equation from the model.

```
* choose a numeraire

W.FX = 1;
```

Now solve the model.

```
SOLVE GE USING MCP;
```

As a counter-factual, double labor productivity and resolve.

```
* double labor productivity


ALPHA = 4;
SOLVE GE USING MCP;
```

Here is the solution to the first solve statement. Notice that W.FX shows up as fixing both the upper and lower bounds for W.

	LOWER	LEVEL	UPPER	MARGINAL
---- VAR P	.	0.500	+INF	.
---- VAR X	.	200.000	+INF	.
---- VAR W	1.000	1.000	1.000	EPS
---- VAR INCOME	.	100.000	+INF	.

Here is the solution to the second solve statement. Doubling of labor productivity allows twice as much X to be produced from the fixed supply of labor.

With the wage rate fixed at W = 1, the equilibrium price of X falls in half: twice as much X can be purchased from the income from a unit of labor.

```
                        LOWER          LEVEL          UPPER        MARGINAL


---- VAR P              .              0.250          +INF            .
---- VAR X              .            400.000          +INF            .
---- VAR W            1.000            1.000          1.000          EPS
---- VAR INCOME         .            100.000          +INF            .
```

In order to appreciate correctly the role of the numeraire, suppose
   instead we wanted to choose X as numeraire.

The first thing that we have to do is "unfix" W. This actually requires
   two separate statements, since .FX involves fixing both the lower
   and upper bounds of W. Then we set P as numeraire.

```
W.UP = +INF;
W.LO = 0;
P.FX = 1;

SOLVE GE USING MCP;


* double labor productivity

ALPHA = 4;
SOLVE GE USING MCP;
```

Here are the solutions for the two runs of our model.

	LOWER	LEVEL	UPPER	MARGINAL
---- VAR P	1.000	1.000	1.000	EPS
---- VAR X	.	200.000	+INF	.
---- VAR W	.	2.000	+INF	.
--- VAR INCOME	.	200.000	+INF	.

	LOWER	LEVEL	UPPER	MARGINAL
---- VAR P	1.000	1.000	1.000	EPS
---- VAR X	.	400.000	+INF	.
---- VAR W	.	4.000	+INF	.
---- VAR INCOME	.	400.000	+INF	.

In comparing the two alternative choices of numeraire, note that the choice does not affect the only "real" variable of welfare significance, the output of X.

The variables affected are "nominal" variables. Alternatively, only relative prices matter and the "real wage", W/P is the same under either choice of numeraire.

```
$TITLE: M1-5, Structure of a general-equilibrium model
* simple (almost trivial) example of a one-good, one-factor,
* one-consumer economy


PARAMETERS
 LBAR      labor supply (fixed and inelastic)
 ALPHA     productivity parameter  X = ALPHA*L;


LBAR = 100;
ALPHA = 2;



NONNEGATIVE VARIABLES
 P          price of X
 X          quantity of X
 W          wage rate
 INCOME     income from labor supply;



EQUATIONS
 ZPROFIT       zeroprofits in X production
 CMKTCLEAR     commodity (X) market clearing
 LMKTCLEAR     labor market clearing
 CONSINCOME    consumer income balance;
```

```
ZPROFIT..     W/ALPHA =G= P;

CMKTCLEAR..  X =G= INCOME/P;

LMKTCLEAR..  LBAR =G= X/ALPHA;

CONSINCOME.. INCOME =G= W*LBAR;

MODEL GE /ZPROFIT.X, CMKTCLEAR.P, LMKTCLEAR.W, CONSINCOME.INCOME/;

* set some starting values

P.L = 1;
W.L = 1;
X.L = 200;
INCOME.L = 100;

* choose a numeraire

W.FX = 1;

OPTION MCP = PATH;
SOLVE GE USING MCP;

* double labor productivity
```

```
ALPHA = 4;

SOLVE GE USING MCP;

* change numeraire

W.UP = +INF;
W.LO = 0;
P.FX = 1;

ALPHA = 2;

SOLVE GE USING MCP;

* double labor productivity

ALPHA = 4;

SOLVE GE USING MCP;




$ontext
formulated as an NLP
the first theorem of welfare economics says that a competitive
equilibrium is Pareto optimal
```

in some very simple situation, such as with a single consumer
this means that equilibrium can also be found as the solution to a
simple NLP: maximizing utility subject to constraints.
```
$offtext

ALPHA = 2;
LBAR = 100;


VARIABLE
  U;


EQUATIONS
  OBJECTIVE;

OBJECTIVE..  U =E= X**0.5;

MODEL GE_NLP / OBJECTIVE, ZPROFIT, CMKTCLEAR, LMKTCLEAR, CONSINCOME/;

SOLVE GE_NLP USING NLP MAXIMIZING U;
```

A general formulation of general-equilibrium models

The purpose of this section is to discuss a general formulation.

First, general-equilibrium models consist of *activities* that transform some goods and factors into others.

These include outputs of goods from inputs, trade activities that transform domestic into foreign goods and vice versa, activities that transform leisure into labor supply, and activities that transform goods into utility (welfare).

Activities are most usefully represented by their dual, or cost-functions.

The conditions for equilibrium: marginal cost for each activity is greater than or equal to price, complementary variable being the equilibrium quantity or "level" that activity.

A quantity variable is complementary to a price equation.   With competition and constant returns to scale, these are also referred to as *zero profit conditions*.

Second, general-equilibrium models consist of *market clearing conditions*.

A *commodity* is a general term that includes goods, factor of production, and even utility. Thus X and labor are both commodities in our example.

Market clearing conditions require that the supply of a commodity is greater than or equal to its demand in equilibrium, with the complementary variable being the price of that commodity.

A price variable is complementary with a quantity equation.

Finally, there are *income-balance equations* for each "agent" in a
   model.

Agents are generally households, but often include a government
   sector, or the owner of a firm in models with imperfect
   competition.

Expenditure (*EXP*) equals income for each agent.

Let i subscript activities, also referred to as production sectors.

Let j subscript commodities, and let k subscript agents or
   households. Our general formulation is then:

Inequality		Complementary variables	No. of unknowns
(1) Zero profits	$MC_i \geq P_i$	Activity (quantity) level i	i
(2) Market clearing	$S_j \geq D_j$	Price of commodity j	j
(3) Income balance	$EXP_k \geq INCOME_k$	Income k	k

The general-equilibrium system then consists of i+j+k inequalities in i+j+k unknowns:

      i activity levels,
      prices of j commodities
      incomes/expenditures of k agents.

As per our earlier discussion, the price level is indeterminate and one price is fixed as a numeraire.

Illustration of gradient method: find the zero of the
excess supply function: $e(p^*) = s(p^*) - d(p^*) = 0$

slope = $e'(p^0)$

$e(p^0)$

$e(p)$
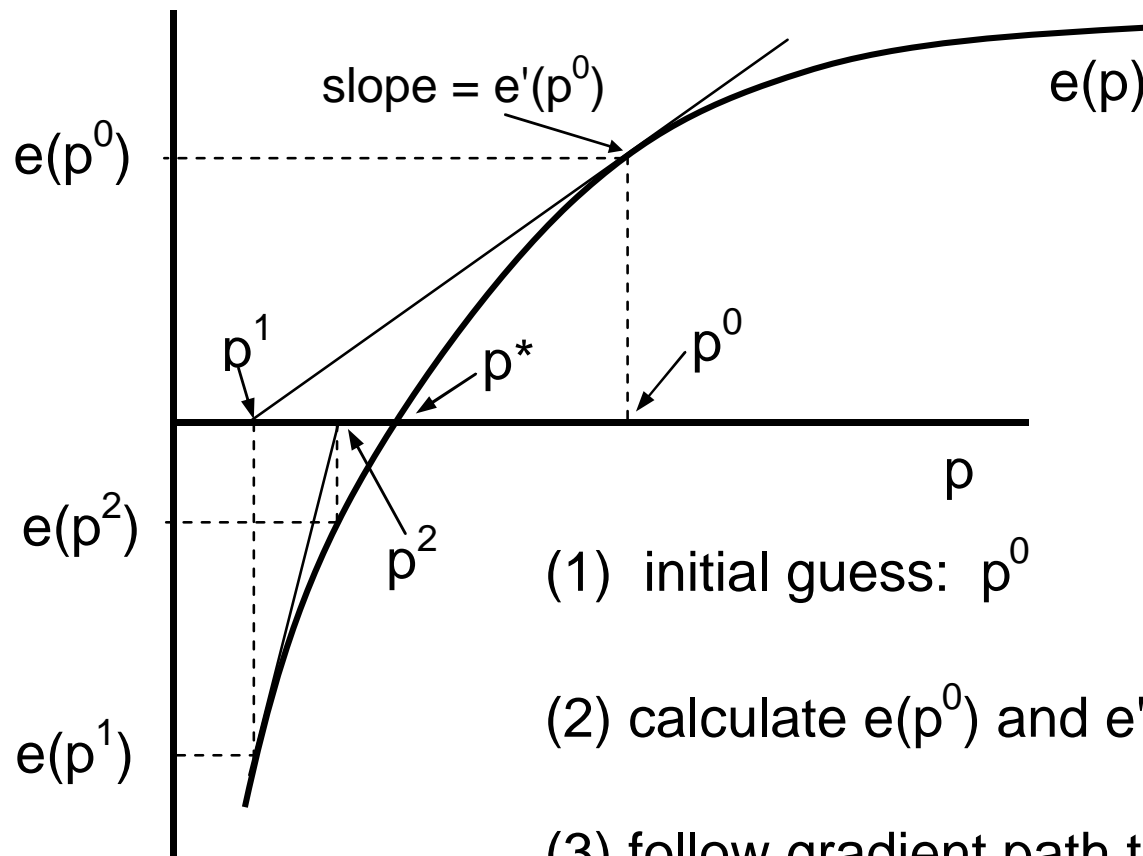
iteration rule:
$$p^{i+1} = p^i - [e'(p^i)]^{-1} e(p^i)$$

p^1

p^*

p^0

p

$e(p^2)$

p^2

(1) initial guess: p^0

(2) calculate $e(p^0)$ and $e'(p^0)$ (exit if $|e(p^0)| < \varepsilon$)

$e(p^1)$

(3) follow gradient path to e = 0

(4) new guess p1 implicitly given by
$$e(p^0)/(p^0 - p^1) = e'(p^0)$$

$$p^{i+1} = p^i - [e'(p^i)]^{-1} e(p^i) \quad \text{(generalizes to nxn)}$$

Figure 7b: Additional volume of trade from allowing trade in A and C (no trade in B)