Econ 4858, Homework 1 - part 1, Professor Martins.

1. Assume X is a random variable with density function

$$f(x) = \begin{cases} 1/(b-a), & \text{if } x \in [a,b] \\ 0, & \text{if } x \notin [a,b] \end{cases},$$

with $a < b$ for $a, b \in \mathbb{R}$.

(a) Obtain $E(X)$ and $V(X)$.

Answer: By definition $E(X) = \int_a^b \frac{x}{b-a} dx = \frac{1}{b-a} |_a^b \frac{x^2}{2} = \frac{1}{b-a}\left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{b+a}{2}$.
By the definition,

$$V(X) = E(X^2) - (E(X))^2 = E(X^2) - (\frac{b+a}{2})^2.$$

But,

$$\begin{aligned} E(X^2) &= \int_a^b \frac{x^2}{b-a} dx = \frac{1}{b-a} |_a^b \frac{x^3}{3} = \frac{1}{b-a}\left(\frac{b^3}{3} - \frac{a^3}{3}\right) \\ &= \frac{1}{3(b-a)}\left((b-a)^3 + 3ab(b-a)\right) = \frac{1}{3}(b-a)^2 + ab. \end{aligned}$$

Substituting back into the expression for variance, we have

$$V(X) = E(X^2) - (E(X))^2 = \frac{1}{3}(b-a)^2 + ab - (\frac{b+a}{2})^2 = \frac{(b-a)^2}{12}.$$

(b) Suppose $a = 0$, $b = 1$ and $Y = 3X - 2$. Find the density function of Y.
Answer: Using results from class slides we have. For $Y = 3X - 2$:

$$f_Y(y) = f_X\left(\frac{y+2}{3}\right)\left|\frac{d\left(\frac{y+2}{3}\right)}{dy}\right| = \frac{1}{3}.$$

2. Let X and Y be random variables (both discrete or continuous). Show that:
a) if c is a constant, show that $E(cX) = cE(X)$; b) if c is a constant, show that $V(cX) = c^2V(X)$ and $V(c) = 0$.

Answer:

a) $E(cX) = \int cxf_X(x)dx = c\int xf_X(x)dx = cE(X)$ where f_X is the density of X.

b) $V(cX) = E((cX)^2) - (E(cX))^2 = c^2E(X^2) - c^2(E(X))^2 = c^2(E(X^2) - E(X)^2) = c^2V(X)$. $V(c) = E(c^2) - E(c)^2 = E(c^2 - c^2) = 0$

3. Suppose there are $N \in \mathbb{N}$ assets that compose a portfolio P. Assume that n_i shares of asset i $(i = 1, \ldots, N)$ are purchased and the price of asset i in time period t is denoted by P_{it}. Let V_t be the value of the portfolio in time period t and $w_{it} = \frac{P_{it}n_i}{V_t}$ be the share of the investment on asset i in period t. Show that net return and gross returns of holding the portfolio for one time period can be written as a weighted average of net returns and gross returns of the component assets of the portfolio. Show that the same is not true for log-returns.

Answer: Net return was done in class. See the last two slides from lecture 1. For gross returns note that

$$\rho_{t+1}^P = \frac{V_{t+1}}{V_t} = \frac{1}{V_t}\sum_{i=1}^n P_{i,t+1}n_i = \frac{1}{V_t}\sum_{i=1}^n (P_{i,t+1}/P_{it})n_i P_{it} = \sum_{i=1}^n \rho_{i,t+1}\frac{n_i P_{it}}{V_t} = \sum_{i=1}^n \rho_{i,t+1}w_{it}.$$

The result doesn't hold for log-returns due to the nonlinearity of the log function.

4. Suppose $Y = \mu X + \sigma Z$ where Z is a random variable with density f_Z, X is a non-stochastic variable, $\sigma > 0$ and $\mu \in \mathbb{R}$ are unknown constants. Obtain the density of Y.

   Answer: Since $Y = \mu X + \sigma Z$ and X and μ are constants, we can write $g(z) = \mu X + \sigma z$ and $g^{-1}(z) = \sigma^{-1}(y - \mu X)$. Hence, $\frac{d}{y}g^{-1}(y) = \frac{1}{\sigma}$. Hence,

$$f_Y(y) = f_Z(\sigma^{-1}(y - \mu X))\frac{1}{\sigma}.$$

5. Suppose $\{X_i\}_{i=1}^n$ is a sequence of independent and identically distributed random variables. Now define,

$$\bar{x}_n = \frac{1}{n}\sum_{i=1}^n X_i.$$

   Obtain $E(\bar{x}_n)$ and $V(\bar{x}_n)$.

   Answer: Since $\{X_i\}_{i=1}^n$ is an IID sequence, if $E(X_i)$ exists it does not depend on i. Thus, let $E(X_i) = \mu$. Similarly, if $V(X_i)$ exists it does not depend on i. Thus, let $V(X_i) = v$. Also, independence implies that the covariance $C(X_i, X_j) = 0$ for any $i \neq j$. Consequently,

$$\begin{aligned} E(\bar{x}_n) &= E\left(\frac{1}{n}\sum_{i=1}^n E(X_i)\right) = \mu \\ V(\bar{x}_n) &= V\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{1}{n^2}\sum_{i=1}^n V(X_i) = \frac{v}{n}. \end{aligned}$$

6. In class we defined log-returns from holding a financial asset for one time period by

$$r_t = \log\frac{P_t}{P_{t-1}}$$

   where P_t is the price of the financial asset in time period t. Assume that $r_t \sim N(\mu, \sigma^2)$ for all $t = 1, \cdots, n$ and $\{r_t\}_{t=1}^n$ forms an independent sequence.

   What is the theoretical probability that the log-return from holding this asset for h time periods will exceed a predetermined value r?

   Answer: $r_{t,h} = \log\frac{P_t}{P_{t-h}} = \sum_{j=0}^{h-1} r_{t-j}$. Hence, $E(r_{t,h}) = h\mu$ and $V(r_{t,h}) = h\sigma^2$. Thus,

$$P(r_{t,h} > r) = 1 - P(r_{t,h} \leq r) = 1 - F_Z\left(\frac{r - h\mu}{\sigma\sqrt{h}}\right)$$

   where F_Z is the distribution function associated with a standard normal density.