Econ 4858, Homework 1 - part 2, Professor Martins.

1. Assume X is a random variable with density function

$$f(x) = \begin{cases} 1/(b-a), & \text{if } x \in [a,b] \\ 0, & \text{if } x \notin [a,b] \end{cases}, \tag{1}$$

with $a < b$ for $a, b \in \mathbb{R}$.

Answer: By definition $E(X) = \int_a^b \frac{x}{b-a} dx = \frac{1}{b-a} |_a^b \frac{x^2}{2} = \frac{1}{b-a}\left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{b+a}{2}$.
By the definition,

$$V(X) = E(X^2) - (E(X))^2 = E(X^2) - (\frac{b+a}{2})^2.$$

But,

$$\begin{aligned} E(X^2) &= \int_a^b \frac{x^2}{b-a} dx = \frac{1}{b-a} |_a^b \frac{x^3}{3} = \frac{1}{b-a}\left(\frac{b^3}{3} - \frac{a^3}{3}\right) \\ &= \frac{1}{3(b-a)}\left((b-a)^3 + 3ab(b-a)\right) = \frac{1}{3}(b-a)^2 + ab. \end{aligned}$$

Substituting back into the expression for variance, we have

$$V(X) = E(X^2) - (E(X))^2 = \frac{1}{3}(b-a)^2 + ab - (\frac{b+a}{2})^2 = \frac{(b-a)^2}{12}.$$

Suppose $a = 0$, $b = 1$ and $Y = X^2$. Find the density of $Y = X^2$. In this case, obtain the 0.75-quantile of Y.

Suppose $a = 1$ and $b = 5$. Obtain the the 0.1 quantile for X and $1/X$.
Answer: $X \sim U_{[1,5]}$ which implies that $f_X(x) = \begin{cases} 1/4 & \text{if } x \in [1,5] \\ 0 & \text{if } x \notin [1,5] \end{cases}$. Then, $F_X(x) = \int_1^x \frac{1}{4} dy = \frac{1}{4}(x-1)$. Let the 0.1-quantile of F_X be denoted by $y_{0.1}$. Then, $0.1 = F_X(x_{0.1}) = \frac{1}{4}(x_{0.1} - 1)$. This implies $x_{0.1} = 1.4$.
For $Y = 1/X$ we have

$$f_Y(y) = f_X(1/y) \left| \frac{d(1/y)}{dy} \right| = \frac{1}{4y^2}.$$

$$F_Y(y_{0.1}) = \int_{1/5}^{y_{0.1}} \frac{1}{4z^2} dz = \frac{1}{4} |_{1/5}^{y_{0.1}} (-1/2) = \frac{1}{4}(-1/y_{0.1} + 5),$$

which implies $y_{0.1} = 0.217$.

2. Let A be a subset of $\mathbb{R}$ and consider the indicator function of the set A, viz.,

$$I_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}, \tag{2}$$

Consider a finite and independent collection $\{X_i\}_{i=1}^n$ of random variables that have the same distribution, say F. Let

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n I_{X_i \le x}$$

Obtain $E(F_n(x))$ and $V(F_n(x))$ for some $x \in \mathbb{R}$.

Answer:

$$\begin{aligned} E(F_n(x)) &= E\left(\frac{1}{n}\sum_{i=1}^{n} I_{X_i \le x}\right) = \frac{1}{n} n E(I_{X_i \le x}) \\ &= 1 \times P(X_i \le x) + 0 \times P(X_i < x) = F(x). \end{aligned}$$

$$\begin{aligned} V(F_n(x)) &= V\left(\frac{1}{n}\sum_{i=1}^{n} I_{X_i \le x}\right) = \frac{1}{n^2} n V(I_{X_i \le x}) \text{ since } \{X_i\}_{i=1}^n \text{ is an independent collection} \\ &= \frac{1}{n}\left(E(I^2_{X_i \le x}) - F(x)^2\right) = \frac{1}{n}\left(E(I_{X_i \le x}) - F(x)^2\right) \\ &= \frac{1}{n}\left(F(x) - F(x)^2\right) = \frac{1}{n} F(x)\left(1 - F(x)\right). \end{aligned}$$

3. Answer problem 2 (items (a) and (b)) from chapter 3 of your textbook.

Answers: 2.
a) $r_t(3) \sim N(0.3, 1.8)$.
b) $P(r_t(3) < 2) = P\left(\frac{r_t(3)-0.3}{\sqrt{1.8}} < \frac{2-0.3}{\sqrt{1.8}}\right) = P\left(Z < \frac{1.7}{\sqrt{1.8}}\right) = 0.8974$.

4. Answer problem 5 (items (a) and (b)) from chapter 3 of your textbook.

Answers: 5.

a) $X_k^2 = X_0^2 \exp(2(r_1 + \cdots + r_k))$ and $E(X_k^2) = X_0^2 E\left(\exp(2r_1 + \cdots + 2r_k)\right)$. Let $\gamma_k = 2r_k$, since $r_k \sim N(\mu, \sigma^2)$ we have $\gamma_k \sim N(2\mu, 4\sigma^2)$. Hence,

$$\begin{aligned} E(X_k^2) &= X_0^2 E(\exp(\gamma_1 + \cdots + \gamma_k)) = X_0^2 E(\exp(2\mu k + 2k\sigma^2)) \\ V(X_k) &= X_0^2 \exp(2\mu k + 2k\sigma^2 - X_0^2 \exp(2\mu k + k\sigma^2) \\ &= X_0^2 \exp(2\mu k)\exp(k\sigma^2)(\exp(k\sigma^2) - 1) \end{aligned}$$

b) $X_1 = g(r_1) = X_0 \exp(r_1)$ implies $\log X_1 = \log X_0 + r_1$ which gives $\log X_1 - \log X_0 = r_1$. So, $g^{-1}(X_1) = \log X_1 - \log X_0$ and $\frac{d}{dX_1} g^{-1}(X_1) = \frac{1}{X_1}$, hence $f_1(X_1) = \phi(\log X_1 - \log X_0)\frac{1}{X_1}$ where $\phi(x)$ is the density for a standard normal.

5. The MATLAB data set AAPL.mat contains six columns. The fifth column contains daily closing price of Apple stock from 01/03/2000 until 07/27/2011.

   (a) Use MATLAB and these price to calculate daily log-returns on Apple stock. Produce a plot of these returns against time.

   (b) Assuming these log-returns are independently drawn and identically distributed as $N(\mu, \sigma^2)$, estimate the parameters of this distribution by the method of moments. Produce an adjusted method of moments estimator for σ^2 that is unbiased.

   (c) Use the estimated parameters to obtain an estimate for the expected price of the stock at each time period in the sample.