Econ 4858, Homework 2 - Part 1, Professor Martins.

1. Answer problems 3 and 13 from chapter 2 of your textbook. On problem 13, also obtain the expected value and the variance of the ML estimator. Recall that the density associated with a random variable with exponential distribution is given by

$$f(x) = \frac{1}{\theta}\exp(-x/\theta) \text{ for } \theta, x > 0.$$

Answers: 3. By definition,

$$\hat{\sigma}^2_{ML} = \underset{\sigma^2\in(0,\infty)}{\text{argmax}} \frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2}\frac{(X_i-\mu)^2}{\sigma^2}\right)\right) \equiv \underset{\sigma^2\in(0,\infty)}{\text{argmax}}\ \ell_n(\sigma^2).$$

Now,

$$\frac{d}{d\sigma^2}\ell_n(\sigma^2) = \frac{1}{n}\sum_{i=1}^{n}\left(-\frac{1}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\frac{(X_i-\mu)^2}{\sigma^2}\right)$$

and solving for $\hat{\sigma}^2_{ML}$ such that $\frac{d}{d\sigma^2}\ell_n(\hat{\sigma}^2_{ML}) = 0$ gives $\hat{\sigma}^2_{ML} = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$.

Answers: 13. The density associated with a random variable with exponential distribution is given by

$$f(x) = \frac{1}{\theta}\exp(-x/\theta) \text{ for } \theta, x > 0.$$

Then, the log-likelihood function is given by

$$\ell_n(\theta) = \underset{\theta\in(0,\infty)}{\text{argmax}} \frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{1}{\theta}\exp(-X_i/\theta)\right).$$

Taking a derivative with respect to θ and solving for $\hat{\theta}_{ML}$ such that $\frac{d}{d\theta}\ell_n(\hat{\theta}_{ML}) = 0$, we have

$$\frac{d}{d\theta}\ell_n(\hat{\theta}_{ML}) = \frac{1}{n}\sum_{i=1}^{n}\left(-\frac{1}{\hat{\theta}_{ML}} + \frac{1}{\hat{\theta}^2_{ML}}X_i\right) = 0$$

which implies $\hat{\theta}_{ML} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

The expected value of $\hat{\theta}_{ML}$ is $E(\hat{\theta}_{ML}) = E(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \theta$. The variance is

$$V(\hat{\theta}_{ML}) = V(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n^2}nV(X_i) = \theta^2/n.$$

2. In class we defined the empirical distribution function $F_n(r)$ associated with a random sample $\{r_t\}_{t=1}^n$; a) Obtain an asymptotically valid $\alpha = 0.95$ confidence interval for $F_r(r)$ the distribution of r_t; b) Use the APPL.mat data in the class website to test the hypothesis that log-return on Apple stock is normally distributed; c) Test the hypothesis that log-return on Apple stock is comes from a location-scale Student-t distribution.

   Answer: a) As derived in class we have that

$$E(F_n(r)) = F_r(r) \text{ and } V(F_n(r)) = \frac{1}{n}(1-F_r(r))F_r(r)$$

Hence, by Lévy's CLT

$$\frac{\sqrt{n}}{(1-F_r(r))F_r(r)}\left(F_n(r)-F_r(r)\right)\to N(0,1).$$

Hence,

$$P\left(-1.96\leq\frac{\sqrt{n}}{(1-F_r(r))F_r(r)}\left(F_n(r)-F_r(r)\right)\leq 1.96\right)=0.95$$

b) Use MLESP500_KS.m with Apple data.

Answer: Directly from the code provided in the website.

c) Use MLE_SP500_student_t.m with Apple data.

Answer: Directly from the code provided in the website.

3. Use the data set AAPL.mat to obtain daily log-returns on Apple stock. Assuming that these data are:

   (a) independently drawn and identically distributed as $Student-t(\mu,\sigma^2,v)$. Estimate the parameters of this distribution by Maximum Likelihood using MATLAB.

Answers: Use codes MLE_SP500.m and MLE_SP500_student_t.m for items (a) and (b), respectively changing the data set used to AAPL.mat.

4. In class we considered, and used, two estimation procedures - the method of moments and the method of maximum likelihood. Suppose that you are interested in a random variable X that has the following density

$$f(x)=\begin{cases}\frac{1}{b-a} & \text{if } x\in[a,b]\\ 0 & \text{if } x\notin[a,b]\end{cases}$$

where $a,b\in\mathbb{R}$ with $b>a$. The values of a and b are unknown and must be estimated using a random sample $\{X_i\}_{i=1}^n$. Obtain the method of moments estimators for a and b. Suppose that instead of the method of moments estimator for b you propose the following estimator,

$$\tilde{b}_n=\max_{1\leq i\leq n}\{X_i\}.$$

What is the probability that $|\tilde{b}_n-b|$ will exceed a certain $\epsilon>0$?

Answer: If the random variable X has density given by f, then $E(X)=\frac{a+b}{2}$ and $V(X)=\frac{(b-a)^2}{12}$. we use these two moments to obtain estimators for a and b. Let $\bar{X}=\frac{1}{n}\sum_{i=1}^n X_i$ and $s^2=\frac{1}{n}\sum_{i=1}^n(X_i-\bar{X})^2$ and define the following equations

$$\bar{X}=\frac{\hat{a}_n+\hat{b}_n}{2} \tag{1}$$

and

$$s^2=\frac{(\hat{b}_n-\hat{a}_n)^2}{12}. \tag{2}$$

Solving equations (1) and (2) for $\hat{a}_n$ and $\hat{b}_n$ we obtain, $\hat{a}_n=\bar{X}-s\sqrt{3}$ and $\hat{b}_n=\bar{X}+s\sqrt{3}$.

$P(|\tilde{b}_n - b| < \epsilon) = P(-(\tilde{b}_n - b) < \epsilon) = P(\tilde{b}_n > b - \epsilon) = 1 - P(\tilde{b}_n \leq b - \epsilon)$. But

$$\begin{aligned} P(\max_{1 \leq i \leq n} \{X_i\} \leq x) &= P(X_1 \leq x, \ldots, X_n \leq x) \\ &= P(X_1 \leq x)P(X_2 \leq x) \cdots P(X_n \leq x), \text{ by independence} \\ &= F(x)^n \text{ by the fact that the distribution is identical for all } X_i \end{aligned}$$

Hence,

$$1 - P(\tilde{b}_n \leq b - \epsilon) = P(|\tilde{b}_n - b| < \epsilon) = 1 - P(|\tilde{b}_n - b| > \epsilon)$$

which implies $P(|\tilde{b}_n - b| > \epsilon) = F^n(b - \epsilon)$. Since F takes values between 0 and 1, as $n \to \infty$ we have $P(|\tilde{b}_n - b| > \epsilon) \to 0$.