Econ 4858, Homework 2 - part 2, Professor Martins.

1. Use Kolmogorov's Law of Large Numbers to show that the method of moments estimators for b that you obtained in question 4 of Homework 2 - part 1 is a consistent estimator.

   Answer: By Kolmogorov's Law of Large Numbers $\bar{X} \xrightarrow{p} \frac{a+b}{2}$ and $s^2 \xrightarrow{p} \frac{(b-a)^2}{12}$. Consequently, $s \xrightarrow{p} \frac{(b-a)}{2\sqrt{3}}$. Hence,

$$\hat{a}_n = \bar{X} - s\sqrt{3} \xrightarrow{p} \frac{a+b}{2} - \frac{(b-a)}{2\sqrt{3}}\sqrt{3} = a$$

   and

$$\hat{b}_n = \bar{X} + s\sqrt{3} \xrightarrow{p} \frac{a+b}{2} + \frac{(b-a)}{2\sqrt{3}}\sqrt{3} = b$$

2. Suppose $\{r_t\}_{t=1}^n$ is a random sample of log-returns on a certain financial asset and assume that $E(r_t) = \mu$ and $V(r_t) = \sigma^2$. a) Using Lèvy's Central Limit Theorem obtain an asymptotically valid $\alpha = 0.95$ confidence interval for μ.

   Answer: a) $\mu_{MM} = \frac{1}{n}\sum_{t=1}^n r_t$ and $E(\mu_{MM}) = \mu$ and $V(\mu_{MM}) = \sigma^2/n$.

   By Lévy's CLT we have that

$$\sqrt{n}\left(\frac{\mu_{MM} - \mu}{\sigma}\right) \xrightarrow{d} N(0,1).$$

   Hence, for $\alpha = 0.95$

$$P\left(-1.96 \leq \sqrt{n}\frac{\mu_{MM} - \mu}{\sigma} \leq 1.96\right) = 0.95$$

   which gives

$$P\left(\mu_{MM} - 1.96\sigma/\sqrt{n} \leq \mu \leq \mu_{MM} + 1.96\sigma/\sqrt{n}\right) = 0.95$$

   Substituting σ with $\sqrt{\sigma_{MM}^2}$ provides the upper and lower bound for the interval.

3. Suppose that log-returns $\{r_t\}_{t=1,2,\cdots}$ on a financial asset follow an AR(1) process given by

$$r_t = \alpha r_{t-1} + \varepsilon_t, \text{ where } \varepsilon_t \sim NID(0, \sigma^2).$$

   Obtain $E(r_t)$, $V(r_t)$ and the autocovariance function of this process.

   Answer: Under the assumption that $|\alpha| < 1$, the process is covariance stationary. Hence, $E(r_t) = E(r_{t-1}) = \mu$. Hence, since $E(\varepsilon_t) = 0$, $\mu = \alpha\mu \iff (1-\alpha)\mu = 0$. Since $\alpha \neq 1$, $\mu = 0$.

   Again, by stationarity and the fact that $\mu = 0$, $V(r_t) = E(r_t^2) = E(\alpha^2 r_{t-1}^2 + \varepsilon_t^2 + 2\varepsilon_t r_{t-1}) = \alpha^2 E(r_{t-1}^2) + \sigma^2$. The last equality follows from the fact that r_{t-1} and ε_t are independent. Then, re-arranging the last equality $V(r_t) = \frac{\sigma^2}{1-\alpha^2}$. From your class notes, the fact that $\mu = 0$, and repeated substitution of lagged values,

$$\gamma(h) = Cov(r_t, r_{t-1}) = E(r_t, r_{t-1}) = E\left(\sum_{j=0}^{\infty}\alpha^j\varepsilon_{t-j}\sum_{j=0}^{\infty}\alpha^j\varepsilon_{t-h-j}\right) = \frac{\sigma^2\alpha^{|h|}}{1-\alpha^2}$$

   for $h = \pm 1, \pm 2, \cdots$

4. Answer problems 1, 2 and 6 from chapter 4 of your textbook.

   Answer: 1. (a) Yes, since $|-0.7| < 1$. (b) $E(Y_t) = 5 - 0.7E(Y_{t-1})$. By stationarity $E(Y_t) = E(Y_{t-1})$, hence $E(Y_t) = 5/1.7$. (c) $V(Y_t) = \frac{\sigma_\epsilon^2}{1-\alpha_1^2} = \frac{2}{1-0.49}$. (d) $Cov(Y_t, Y_{t\pm h}) = \frac{\sigma_\epsilon^2}{1-\alpha_1^2}\alpha_1^{|h|} = \frac{2(-0.7)^{|h|}}{1-0.49}$.

2. (a) $V(Y_t) = \frac{2}{1-0.09}$. (b) $Cov(Y_1, Y_3) = \frac{\sigma_\epsilon^2}{1-\phi^2}\phi^2 = \frac{2}{1-0.09}0.3^2$. (c) $V((Y_1+Y_3)/2) = V(Y_1/2) + V(Y_3/2) + 2Cov(Y_1/2, Y_3/2) = \frac{1}{2}\frac{\sigma_\epsilon^2}{1-\alpha_1^2} + \frac{1}{2}Cov(Y_1, Y_3) = \frac{1}{2}\frac{\sigma_\epsilon^2}{1-\alpha_1^2} + \frac{1}{2}\frac{\sigma_\epsilon^2}{1-\phi^2}\phi^2$.

6. (a) Multiplying $Y_t - \mu$ by $Y_{t-k} - \mu$ we have

$$(Y_t - \mu)(Y_{t-k} - \mu) = \phi_1(Y_{t-k} - \mu)(Y_{t-1} - \mu) + \phi_2(Y_{t-k} - \mu)(Y_{t-2} - \mu) + (Y_{t-k} - \mu)\epsilon_t. \quad (1)$$

Taking expectations on both sides gives $\gamma(k) = \phi_1\gamma(k-1) + \phi_2\gamma(k-2)$. Dividing this last equation by $\gamma(0)$ we have

$$\rho(k) = \phi_1\rho(k-1) + \phi_2\rho(k-2). \quad (2)$$

(b) Now, evaluating (2) at $k = 1, 2$ and noting that $\rho(0) = 1$ we have

$$\rho(1) = \phi_1 + \phi_2\rho(1) \text{ and } \rho(2) = \phi_1\rho(1) + \phi_2.$$

Writing these two equations in matrix format completes the proof.

(c) Substituting the given values of $\rho(1)$ and $\rho(2)$ we have $\phi_1 = 0.38$ and $\phi_2 = 0.0476$. Now, $\rho(3) = 0.38 \times 0.2 + 0.0476 \times 0.4$.

5. Suppose that the log-returns on a financial asset are generated by an AR(1) process given by

$$r_t = \alpha r_{t-1} + \varepsilon_t, \text{ where } \varepsilon_t \sim NID(0, \sigma^2). \quad (3)$$

with $|\alpha| < 1$.

(a) Given a sample $\{r_t\}_{t=1}^n$ of size n, derive the least squares estimator for α.
Answer: The least squares estimator for α is given by

$$\hat{\alpha} = \text{argmin}_\alpha S_n(\alpha) = \text{argmin}_\alpha \sum_{t=2}^{n}(r_t - \alpha r_{t-1})^2.$$

Taking the derivative of $S_n(\alpha)$ with respect to α and setting it equal to zero gives,

$$2\sum_{t=2}^{n}(r_t - \hat{\alpha} r_{t-1})(-r_{t-1}) = 0,$$

which imples $\hat{\alpha} = \left(\sum_{t=2}^n r_{t-1}^2\right)^{-1}\sum_{t=2}^n r_{t-1}r_t$.

(b) Obtain the expected value and the variance of the estimator you derived in item a).

Answer: Obtained in class. $E(\hat{\alpha}) = \alpha$ and $V(\hat{\alpha}|r_1, \cdots, r_{n-1}) = \sigma^2\frac{1}{\sum_{t=2}^n r_{t-1}^2}$. Also, $V(\hat{\alpha}) = \sigma^2 E\left(\frac{1}{\sum_{t=2}^n r_{t-1}^2}\right)$

(c) Is the method of moments estimator for α the same as the least squares estimator for α? Prove. Did you have to use normality in your proof?

Answer: Yes. Note that $r_t r_{t-1} = \alpha r_{t-1}^2 + r_{t-1}\varepsilon_t$ and

$$E(r_t r_{t-1}) = \alpha E(r_{t-1}^2) + E(r_{t-1}\varepsilon_t).$$

Since, $E(r_{t-1}\varepsilon_t) = E(E(r_{t-1}\varepsilon_t|r_{t-1})) = 0$ we have

$$\alpha = \frac{E(r_t r_{t-1})}{E(r_{t-1}^2)}.$$

Hence, the method of moments estimator for α is

$$\alpha_{MM} = \frac{\frac{1}{n}\sum_{t=2}^{n} r_{t-1} r_t}{\frac{1}{n}\sum_{t=2}^{n} r_{t-1}^2}$$

and $\hat{\alpha} = \alpha_{MM}$. No normality is used.