Econ 4858, Homework 3, Professor Martins.

Question 1: Let $\{r_t\}_{t=1,2,\cdots}$ be a sequence of log-returns on a financial asset.

1. What conditions characterize *covariance stationary* of this sequence?
2. What conditions characterize *strict stationary* of this sequence?
3. Does strict stationarity imply covariance stationarity? Why? Why not? Explain using mathematical arguments.
4. Suppose you assume that $r_t = \phi_0 + \phi r_{t-2} + \varepsilon_t$ with $\varepsilon_t \sim NIID(0, \sigma^2)$ (independent and identically distributed as a normal random variable).

   a) Assuming that $|\phi| < 1$ show that $r_t = \frac{\phi_0}{1-\phi} + \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-2j}$.

   b) Under the assumption in a), obtain $V(r_t)$, $Cov(r_t, r_{t-1})$ and $Cov(r_t, r_{t-2})$.

   c) Is the the covariance structure of this process the same is that for an AR(1) process? If not, what is the difference? Is this process covariance stationary?

   d) Now, suppose that $\phi_0 = 0$. Derive the least squares estimator for ϕ given a sample $\{r_{-1}, r_0, r_1, \cdots, r_n\}$ and denote it by $\hat{\phi}$. What is $E(\hat{\phi}|r_{-1}, r_0, r_1, \cdots, r_n)$ given that $E(r_{t-2}\varepsilon_t) = 0$? What is $E(\hat{\phi})$?

   e) Obtain $V(\hat{\phi}|r_{-1}, r_0, r_1, \cdots, r_n)$.

Question 2: Consider the following covariance stationary AR(2) process

$$r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \varepsilon_t$$

with $\varepsilon_t \sim NIID(0, \sigma^2)$. Assume that you observe a sample $\{r_t\}_{t=-1,0,1,\cdots,n}$ and let

$$\vec{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}, \; R = \begin{pmatrix} 1 & r_0 & r_{-1} \\ 1 & r_1 & r_0 \\ \vdots & \vdots & \vdots \\ 1 & r_{n-1} & r_{n-2} \end{pmatrix} \text{ and } \phi = \begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix}.$$

In class, we showed that the least squares estimator for ϕ, denoted by $\hat{\phi}$, is given by $\hat{\phi} = (R^T R)^{-1} R^T \vec{r}$. We also showed that

$$\sqrt{n}(\hat{\phi} - \phi) \xrightarrow{d} N(0, \sigma^2 Q^{-1}),$$

where Q is the matrix such that $\frac{1}{n} R^T R \xrightarrow{p} Q$.

1. What are the dimensions of $\sigma^2 Q^{-1}$? What do its elements represent?
2. Provide a consistent estimator for $\sigma^2 Q^{-1}$.
3. How would you test the hypothesis that $\phi_1 = \phi_2 = 0$? Write down the test statistic, give its distribution and carefully describe the testing procedure, including the rule for rejecting and failing to reject the hypothesis.

4. Consider the following lines of a MATLAB code written to estimate the AR(2) process in this question. The lines are consecutively numbered from 1 to 22.

```
1 load AAPL.mat;
2 dd = AAPL;
3 n=size(dd,1);
4 sp = dd(:,5);
5 sp0 = sp(2:n,1);
6 sp1 = sp(1:n-1,1);
7 rg = log (sp0./sp1);
8 r2 = rg(1:n-3);
9 r1 = rg(2:n-2);
10 r = rg(3:n-1);
11 bfone = ones(size(r,1),1);
12 R= [bfone r1 r2];
13 phihat = inv(R' * R) *(R' * r);
14 sigma_hat2 = (1/size(r,1))*(r-R*phihat)' *(r-R*phihat);
15 var_phi = (1/size(r,1))*(R' *R);
16 var_phi = sigma_hat2*var_phi;
17 R1 = [1 0];
18 g = 0;
19 aux_=inv(R1*var_phi*R1')*(R1*phihat-g);
20 test_st=size(r,1)*(R1*phihat-g)'*aux_;
21 critical3 = chi2inv(0.95,2);
22 [test_st critical3]
```

a) On line 22 a test statistic and a critical value are arranged in a 1 by 2 vector. Are they correct for testing the hypothesis that $\phi_2 = 0$ at the 0.05 level? If not, what and how the lines on the code must be changed?; b) What is line seven calculating? Explain why sp0 and sp1 have the structure given on lines 5 and 6; c) Why does the vector $r1$ on line 9 start at 2 and ends at $n-2$? Is this correct?

Question 3: The file AAPL.MAT contains a MATLAB data set whose 5th column contains daily closing prices for Apple stock from 01.03.2000 to 07.27.2011. Using these data generate a sequence of log-returns r_t.

1. Estimate the parameters α_0, α_1, σ^2 of the following model

$$r_t = \alpha_0 + \alpha_1 r_{t-1} + \epsilon_t \text{ where } \epsilon_t \sim NIID(0, \sigma^2),$$

using a least squares estimator.

2. Can you reject the hypothesis that $\alpha_0 = 0$ at the 1 percent significance level?

3. Can you reject the hypothesis that $\alpha_1 = 0$ at the 1 percent significance level?

4. Obtain estimates for the $Cov(r_t, r_{t-h})$ when $h = 1, 2, 3$.

Question 4: Now, suppose that

$$r_t = \alpha_0 + \alpha_1 r_{t-1} + \alpha_2 r_{t-1} + \epsilon_t \text{ where } \epsilon_t \sim NIID(0, \sigma^2).$$

1. Estimate the parameters α_0, α_1, α_2, σ^2 using a least squares estimator.

2. Test the hypothesis that $\alpha_1 = \alpha_2 = 0$ at the 5 percent significance level.

3. Test the hypothesis that $\alpha_2 = 0$ at the 5 percent significance level.

4. Produce a plot with with the least squares forecasts $\hat{r}_t$ on the vertical axis and the true values r_t on the horizontal axis. What is the empirical correlation between these two variables?