Econ 4858, Homework 4, Professor Martins-Filho.

1. Suppose $\varepsilon_t \sim IID(0,1)$, that $a_t = \left(1 + 0.5a_{t-1}^2\right)^{1/2} \varepsilon_t$ and that $r_t = 3 + 0.7r_{t-1} + a_t$.
   (a) Find $E(r_t)$.
   (b) Find $V(r_t)$.
   (c) Find the autocovariance function of r_t.
   (d) Find the autocovariance function of a_t^2.

2. Let $\varepsilon_t \sim IID(0,1)$, that $a_t = \left(\alpha_0 + \alpha_1 a_{t-1}^2\right)^{1/2} \varepsilon_t$ and that $r_t - \mu = \phi(r_{t-1} - \mu) + a_t$. Suppose $\mu = 0.7$, $\phi = 0.5$, $\alpha_0 = 1$ and $\alpha_1 = 0.3$.
   (a) Find $E(r_2|r_1 = 1, r_0 = 0.2)$.
   (b) Find $V(r_2|r_1 = 1, r_0 = 0.2)$.

3. Use the data set `AAPL.mat` to obtain a sequence of log-returns and use them for the following questions.

   a) Estimate the following GARCH models:

   (a) $r_t = (\alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 h_{t-1})^{1/2}\varepsilon_t$ where $\varepsilon_t \sim NIID(0,1)$
   (b) $r_t = b_0 + b_1 r_{t-1} + (\alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 h_{t-1})^{1/2}\varepsilon_t$ where $\varepsilon_t \sim NIID(0,1)$.
   (c) $r_t = b_0 + b_1 r_{t-1} + b_2 r_{t-2} + (\alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 h_{t-1} + \beta_2 h_{t-2})^{1/2}\varepsilon_t$ where $\varepsilon_t \sim IID - St(v)$ (Student-t with v degrees of freedom).

   b) For each of these models test the hypothesis that $\beta_1 = 0$ at the 5 percent level of significance.

   c) For model (c) test the hypothesis that $b_1 = 0$ at the 10 percent level of significance.

4. Generate data using the following GARCH model:

$$r_t = \mu + (\alpha_0 + \alpha_1(r_{t-1} - \mu)^2 + \alpha_2(r_{t-2} - \mu)^2 + \beta_1 h_{t-1} + \beta_2 h_{t-2})^{1/2}\varepsilon_t$$

   where $\varepsilon_t \sim IID - St(v)$ (Student-t with v degrees of freedom) with the parameters given by $\mu = 2; \alpha_0 = 1; \beta_1 = 0.3; \beta_2 = 0.1; \alpha_1 = 0.2; \alpha_2 = 0.1; v = 5$.

   a) Use 72000 observations to estimate the parameters of the model assuming $\varepsilon_t \sim N(0,1)$.

   b) Use 72000 observations to estimate the parameters of the model assuming $\varepsilon_t \sim IID - St(v)$.

5. Suppose there are two risky assets (A_1 and A_2) and a risk free asset (F). We know from portfolio theory that the optimal portfolio for an expected targeted return μ_p is a linear combination of the risk free asset and the tangency portfolio. That is,

$$\mu_p = \sum_{i=1}^{2} w_i \mu_i + (1 - w_1 - w_2)\mu_F \qquad (1)$$

where w_1 and w_2 are weights associated with risky assets A_1 and A_2, μ_1 and μ_2 are expected returns on assets A_1 and A_2 and μ_F is the return on the risk free asset.

a) Obtain w_1 and w_2 that minimize the portfolio risk for the targeted return μ_p.

b) What should be the weight on the risk free asset?

c) What are the weights associated with the tangency portfolio?

d) The weights in item a) depend on μ_1, μ_2 but also on the variances of the returns for A_1 and A_2 and their covariances. Suppose you have two time series on these returns, i.e., $\{R_{tA_1}\}_{t=1}^{n}$ and $\{R_{tA_2}\}_{t=1}^{n}$. Propose estimators for μ_1, μ_2, the variances and covariance of the returns for the risky assets under the assumption that the two time series form iid collections of random variables.

e) Is the estimator for μ_1 that you proposed in d) unbiased?