Econ 4858 - Financial Econometrics, Professor Martins
Midterm Examination 1
Date: September 28, 2023.

IMPORTANT INSTRUCTIONS: The pages of your set of answers should be numbered consecutively starting at 1. Write only on one-side of each sheet of paper. Start the answer for a new question on a new sheet of paper. Answer questions 1 or 2, and 3.

Question 1: Let $\{P_t\}_{t=0,1,\cdots}$ be a sequence of prices for a given financial asset.

a) Define gross returns, net returns and log returns from holding the asset from period $t-1$ to period t. What are the range of values that each of these concepts of returns take?

Answer: (3 points) Gross returns are defined as $R_t = \frac{P_t}{P_{t-1}}$ and take values in $[0, \infty)$. Net returns are defined as $N_t = \frac{P_t - P_{t-1}}{P_{t-1}}$ and take values in $[-1, \infty)$. log-returns are defined as $\log R_t = \log P_t - \log P_{t-1}$ and take values in $(-\infty, \infty)$.

b) If log-returns from holding a financial asset for h periods at time t is denoted by $r_{t,h}$, show that $r_{t,h} = \sum_{j=1}^{h} r_{t-(j-1),1}$.

Answer: (2 points) By definition of $r_{t,h}$ we have

$$r_{t,h} = \log \frac{P_t}{P_{t-h}} = \log \left(\frac{P_t}{P_{t-1}} \frac{P_{t-1}}{P_{t-2}} \cdots \frac{P_{t-(h-1)}}{P_{t-h}} \right) = \log \frac{P_t}{P_{t-1}} + \log \frac{P_{t-1}}{P_{t-2}} + \cdots + \log \frac{P_{t-(h-1)}}{P_{t-h}}$$
$$= r_{t,1} + r_{t-1,1} + \cdots + r_{t-(h-1),1} = \sum_{j=1}^{h} r_{t-(j-1),1}$$

c) In class, for a non-stochastic starting price level P_0 and 1-period log returns $r_{t,1} := r_t$ we have written

$$\log\, P_t = \log\, P_0 + r_t + r_{t-1} + \cdots + r_1 = \log\, P_0 + \sum_{i=1}^{t} r_i$$

where each $r_t \sim N(\mu, \sigma^2)$ and the sequence $\{r_t\}$ is independent. What is the covariance between $\log\, P_t$ and $\log\, P_{t-1}$? Show all your work.

Hint: Recall that the covariance between any two random variables X and Y, denoted by $C(X,Y) = E\left((X - E(X))(Y - E(Y))\right)$.

Answer: (3 points) First note that $E(\log P_t) = \log P_0 + t\mu$ and $\log P_t - E(\log P_t) = \sum_{i=1}^{t} r_i - t\mu$. Now, by definition

$$\begin{aligned}
Cov(\log P_t, \log P_{t-1}) &= E\left(\sum_{i=1}^{t} r_i - t\mu\right)\left(\sum_{i=1}^{t-1} r_i - (t-1)\mu\right) \\
&= E\left(\sum_{i=1}^{t} r_i \sum_{i=1}^{t-1} r_i\right) - (t-1)t\mu^2 \\
&= (t-1)\sigma^2 + t(t-1)\mu^2 - (t-1)t\mu^2 \text{ by independence of } r_i \text{ and } E(r_i^2) = \sigma^2 + \mu^2 \\
&= (t-1)\sigma^2
\end{aligned}$$

Question 2: Suppose $\{r_t\}_{t=1,2,\cdots}$ is a sequence of independent and identically distributed 1-period log-returns on a certain financial asset. Assume that $E(r_t) = \mu$ for all t and that $V(r_t) = \sigma^2$.

a) What is the method of moments estimator for μ? Label this estimator μ_M. Is it an unbiased estimator for μ? Prove. What does it mean for any estimator to be unbiased?

b) What is the variance of μ_M in item a)?

c) Is μ_M a consistent estimator for μ? Explain.

d) What is the the distribution of μ_M if $r_t \sim N(\mu, \sigma^2)$? Provide a 0.95 confidence interval for μ. Are the lower and upper bounds in this interval known? If not, how would you estimate them?

Question 3: In class, and on your homework, we considered several cases where we had to obtain the density of the function of a random variable. Here, consider a continuous random variable X with density function f_X and distribution function F_X, and the random variable Y, which is a function of X given by $Y = g(X) = X^2$.

a) For $y > 0$, show that $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

Answer: By definition $F_Y(y) := P(Y \leq y)$. Hence,

$$F_Y(y) := P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

b) For $y > 0$, show that $f_Y(y) = \frac{1}{2\sqrt{y}} \left(f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right)$.

Answer: The density of Y, denoted by f_Y is given by

$$\begin{aligned} f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \left(F_X(\sqrt{y}) - F_X(-\sqrt{y}) \right) &= f_X(\sqrt{y}) \frac{d}{dy} \sqrt{y} - f_X(-\sqrt{y}) \frac{d}{dy} (-\sqrt{y}) \\ &= f_X(\sqrt{y}) \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{y}} \left(f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right). \end{aligned}$$

c) If X has a standard normal density, i.e., $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-0.5x^2\right)$ obtain the formula for the density of Y.

Answer: Substitute $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-0.5x^2\right)$ in the formula given in b) to obtain

$$\begin{aligned} f_Y(y) &= \frac{1}{2\sqrt{y}} \left(\frac{1}{\sqrt{2\pi}} \exp\left(-0.5y\right) + \frac{1}{\sqrt{2\pi}} \exp\left(-0.5y\right) \right) \\ &= \frac{1}{\sqrt{2\pi y}} \exp\left(-0.5y\right). \end{aligned}$$

Hint: Recall that for any random variable X, $F_X(x) = P(X \leq x)$. If X is continuous and has density f_X, then $F_X(x) = \int_{-\infty}^{x} f_X(u)du$ and $\frac{d}{dx} F_X(x) = f_X(x)$. Also, by the Chain rule, if h and g are differentiable functions and $\frac{d}{dx} h(g(x)) = \frac{d}{dg(x)} h(g(x)) \frac{d}{dx} g(x)$.