Econ 4858 - Financial Econometrics, Professor Martins
Midterm Examination 2
Date: November 2, 2023.

IMPORTANT INSTRUCTIONS: The pages of your set of answers should be numbered consecutively starting at 1. Write only on one-side of each sheet of paper. Start the answer for a new question on a new sheet of paper. Answer ALL questions.

Question 1: Suppose that instead of modeling log-returns associated with a financial asset, you are interested in modeling gross returns from holding the financial asset for one time period. That is, your interest is on the random variable

$$G_t = \frac{P_t}{P_{t-1}}$$

where $P_t > 0$ is the price of the financial asset at time period t.

1. (1 point) Does it make sense to assume that G_t has a Gaussian (Normal) density? Explain.

   Answer. No. G_t takes values in $(0, \infty)$ whereas a normally distributed random variable takes values in $(-\infty, \infty)$.

2. (2 points) Suppose you observe a random sample $\{G_t\}_{t=1}^n$ of gross returns, i.e., a collection of independent and identically distributed random variables, such that G_t has density given by

$$f(x;\beta) = \frac{1}{c\beta^2} x \exp(-x/\beta) \text{ for } 0 < x < \infty$$

   where c is a known constant and $\beta > 0$ is a parameter to be estimated. Obtain the log-likelihood function for the sample $\{G_t\}_{t=1}^n$.

   Answer. From the functional form of the density

$$\log f(G_t;\beta) = -\log c - 2\log\beta + \log G_t - \frac{1}{\beta} G_t$$

   and consequently

$$\ell_n(\beta) := \sum_{t=1}^n \log f(G_t;\beta) = -n\log c - 2n\log\beta + \sum_{t=1}^n \log G_t - \frac{1}{\beta}\sum_{t=1}^n G_t.$$

3. (3 points) Using the log-likelihood function from part 2, obtain the maximum likelihood estimator for the parameter β, call it β_{ML}.

   Answer. Taking the derivative of $\ell_n(\beta)$ with respect to β we obtain

$$\frac{d}{d\beta}\ell_n(\beta) = -2n\beta^{-1} + \frac{1}{\beta^2}\sum_{t=1}^n G_t.$$

   Solving for the value β_{ML} of β that satisfies $\frac{d}{d\beta}\ell_n(\beta_{ML}) = 0$ we obtain

$$\beta_{ML} = \frac{1}{2}\frac{\sum_{t=1}^n G_t}{n}$$

4. (2 points) If $E(G_t) = 2\beta$ given the density in part 2, is β_{ML} equal to the method of moments estimator for β? Prove.

   Answer. If $E(G_t) = 2\beta$, we have that the method of moments estimator for β, say β_{MM} satisfies

$$\frac{\sum_{t=1}^{n} G_t}{n} = 2\beta_{MM} \text{ and consequently } \beta_{MM} = \beta_{ML}.$$

5. (2 points) Is β_{ML} a consistent estimator for β? Prove. Hint: use Kolmogorov's LLN.

   Answer. By Kolmogorov's LLN $\frac{\sum_{t=1}^{n} G_t}{n} \xrightarrow{p} E(G_t) = 2\beta$. Consequently, $\beta_{ML} = \frac{1}{2}\frac{\sum_{t=1}^{n} G_t}{n} \xrightarrow{p} \beta$.

6. (3 points) If $V(G_t) = 2\beta^2$ given the density in part 2, obtain the asymptotic distribution of $\sqrt{n}(\beta_{ML} - \beta)$. Hint: use Lévy's CLT.

   Answer. By Lévy's CLT

$$\frac{\frac{1}{n}\sum_{t=1}^{n} G_t - E(G_t)}{\sqrt{V\left(\frac{1}{n}\sum_{t=1}^{n} G_t\right)}} = \frac{2\beta_{ML} - 2\beta}{\sqrt{2\beta^2/n}} = \frac{\sqrt{n}(\beta_{ML} - \beta)}{\beta/\sqrt{2}} \xrightarrow{d} \mathcal{Z} \sim N(0,1).$$

   Consequently, $\sqrt{n}(\beta_{ML} - \beta) \xrightarrow{d} N(0, \beta^2/2)$.

Question 2: Suppose that log returns r_t on a financial asset follow an $AR(1)$ process given by

$$r_t = \alpha r_{t-1} + u_t \text{ for } t = 2, ..., n. \tag{1}$$

where $\{u_t\}$ is an independent sequence with common distribution $N(0, \sigma^2)$.

1. (3 points) Give conditions for weak (covariance) stationarity of the process. Explain why the conditions you gave guarantee weak stationarity.

   Answer. After m substitution of the lag valued we have $r_t = \alpha^{m+1} r_{t-(m+1)} + \sum_{j=0}^{m} \alpha^j u_{t-j}$. If $|\alpha| < 1$ and letting $m \to \infty$ we have

$$r_t = \sum_{j=0}^{\infty} \alpha^j u_{t-j}. \tag{2}$$

   Stationarity follows from the requirement that $|\alpha| < 1$. It is necessary to obtain the representation in (2).

2. (2 points) Obtain $E(r_t)$, $V(r_t)$ and $Cov(r_t, r_{t+h})$ for $h = \pm 1, \pm 2, \cdots$. What is the correlation between r_t and r_{t+h} for $h = \pm 1, \pm 2, \cdots$.

   Answer. From (2) and the assumption that $u_t \sim N(0, \sigma^2)$ and independent, we immediately obtain $E(r_t) = 0$, $V(r_t) = \sigma^2(1-\alpha^2)^{-1}$ and $Cov(r_t, r_{t+h}) = E(r_t r_{t+h}) = \sigma^2 \alpha^{|h|}(1-\alpha^2)^{-1}$. By definition,

$$Corr(r_t, r_{t+h}) = \frac{Cov(r_t, r_{t+h})}{\sqrt{V(r_t)}\sqrt{V(r_{t-1})}} = \frac{\sigma^2 \alpha^{|h|}(1-\alpha^2)^{-1}}{\sigma^2(1-\alpha^2)^{-1}} = \alpha^{|h|}$$

3. (2 points) Suppose $\alpha \neq 0$. Does this imply that knowledge of r_{t-1} is useful in forecasting the expected value of r_t? Why? Would your answer be different if $\alpha = 0$? Why? Explain.

   Answer. It is useful in forecasting the *conditional* expectation of r_t, since $E(r_t | r_{t-1}) = \alpha r_{t-1}$. Yes, since if $\alpha = 0$ we have $r_t = u_t$, which is independent and identically distributed as a normal random variable.

Question 3: (4 points) In trying to assess whether or not a sequence of log-returns $\{r_t\}_{t=1}^n$ was such that $r_t \sim N(\mu, \sigma^2)$ for all t we used a graphical diagnostic tool called a QQ-plot. What is measured on the horizontal and vertical axis of this plot? What do you look for in a QQ-plot to find "evidence" that $r_t \sim N(\mu, \sigma^2)$? What mathematical argument supports your answer?

Answer. (2 points) On the horizontal axis we measured the quantiles associated with a standard normal distribution, and on the vertical axis we measured the quantiles associated with the empirical distribution of $\{r_t\}_{t=1}^n$. Since, under the assumption that $r_t \sim N(\mu, \sigma^2)$ we have that

$$r_t = \mu + \sigma Z_t \text{ where } Z_t \sim N(0, 1),$$

then for any $p \in (0, 1)$

$$F_{r_t}^{-1}(p) = \mu + \sigma F_{Z_t}^{-1}(p).$$

Hence the quantiles of r_t are a linear function of the quantiles of a standard normal. If we estimate $F_{r_t}^{-1}(p)$ with the quantiles of the empirical distribution of the sample, we should have

$$F_n^{-1}(p) \approx \mu + \sigma F_{Z_t}^{-1}(p).$$

Deviations from a straight line should be evidence of non-normality.