Econ 7818, Homework 1 - part 1, Professor Martins. Due date will be announced later, during class.

1. Let $\mathbb{X}$ be an arbitrary set and consider the collection of all subsets of $\mathbb{X}$ that are countable or have countable complements. Show that this collection is a σ-algebra. Use this fact to obtain the σ-algebra generated by $\mathcal{C} = \{\{x\} : x \in \mathbb{R}\}$.

   Answer: (4 points) Let $\mathcal{F} = \{A \subseteq \mathbb{X} : \#A \leq \#\mathbb{N} \text{ or } \#A^c \leq \#\mathbb{N}\}$, where $\#$ indicates cardinality. First, note that $\mathbb{X} \in \mathcal{F}$ since $\mathbb{X}^c = \emptyset$, which is countable. Second, if $A \in \mathcal{F}$ then either $A = (A^c)^c$ or A^c are countable. That is, $A^c \in \mathcal{F}$. Third, if $A_n \in \mathcal{F}$ for $n \in \mathbb{N}$ we have two possible cases - A_n are all countable, or at least one of these sets is uncountable, say A_{n_0}. For the first case, $\underset{n\in\mathbb{N}}{\cup} A_n$ is the countable union of countable sets, hence it is countable and consequently in $\mathcal{F}$. For the second case, since A_{n_0} is uncountable and in $\mathcal{F}$, it must be that $A^c_{n_0}$ is countable. Also,

$$\left(\underset{n\in\mathbb{N}}{\cup} A_n\right)^c = \underset{n\in\mathbb{N}}{\cap} A^c_n \subset A^c_{n_0}.$$

   Since subsets of countable sets are countable, $\left(\underset{n\in\mathbb{N}}{\cup} A_n\right)^c$ is countable, and consequently $\underset{n\in\mathbb{N}}{\cup} A_n \in \mathcal{F}$.

   Now, let $\mathcal{F}$ be the σ-algebra defined above. Since $\mathcal{C} \subseteq \mathcal{F}$, $\sigma(\mathcal{C}) \subseteq \mathcal{F}$. Also, if $A \in \mathcal{F}$ either A or A^c is countable. Without loss of generality, suppose A is countable. Then, $A = \underset{x\in C}{\cup} \{x\}$ where C is a countable collection of real numbers. Hence, $A \in \sigma(\mathcal{C})$. Hence, $\mathcal{F} \subseteq \sigma(\mathcal{C})$. Combining the two set containments we have $\sigma(\mathcal{C}) = \mathcal{F}$.

2. Denote by $B(x, r)$ an open ball in $\mathbb{R}^n$ centered at x and with radius r. Show that the Borel sets are generated by the collection $B = \{B_r(x) : x \in \mathbb{R}^n,\ r > 0\}$.

   Answer: (3 points) Let $B' = \{B_r(x) : x \in \mathbb{Q}^n,\ r \in \mathbb{Q}^+\}$. Then, $B' \subset B \subset \mathcal{O}_{\mathbb{R}^n}$ and $\sigma(B') \subset \sigma(B) \subset \sigma(\mathcal{O}_{\mathbb{R}^n})$.

   Now, let $S = \underset{B\in B',\, B\subset O}{\bigcup} B$. By construction $x \in S \implies x \in O$. Now, suppose $x \in O$. Then, since O is open, there exists $B(x, \epsilon)$ such that $B(x, \epsilon) \subset O$ where ϵ is a rational number. Since $\mathbb{Q}^n$ is a dense subset of $\mathbb{R}^n$, we can find $q \in \mathbb{Q}^n$ such that $\|x - q\| \leq \epsilon/2$. Consequently,

$$B(q, \epsilon/2) \subset B(x, \epsilon) \subset O.$$

   Hence, $O \subset S$. Thus, every open O can be written as $O = \underset{B\in B',\, B\subset O}{\bigcup} B$. Since B' is a collection of balls with rational radius and rational centers, B' is countable. Thus,

$$\mathcal{O}_{\mathbb{R}^n} \subset \sigma(B') \implies \sigma(\mathcal{O}_{\mathbb{R}^n}) \subset \sigma(B').$$

   Combining this set containment with $\sigma(B') \subset \sigma(B) \subset \sigma(\mathcal{O}_{\mathbb{R}^n})$ completes the proof.

3. Let $(\Omega, \mathcal{F})$ be a measurable space. Show that: a) if μ_1 and μ_2 are measures on $(\Omega, \mathcal{F})$, then $\mu_c(F) := c_1\mu_1(F) + c_2\mu_2(F)$ for $F \in \mathcal{F}$ and all $c_1, c_2 \geq 0$ is a measure; b) if $\{\mu_i\}_{i\in\mathbb{N}}$ are measures on $(\Omega, \mathcal{F})$ and $\{\alpha_i\}_{i\in\mathbb{N}}$ is a sequence of positive numbers, then $\mu_\infty(F) = \sum_{i\in\mathbb{N}} \alpha_i\mu_i(F)$ for $F \in \mathcal{F}$ is a measure.

   Answer: a) (2 points) First, note that $\mu_c : \mathcal{F} \to [0, \infty]$ since $c_1, c_2, \mu_1(F), \mu_2(F) \geq 0$ for all $F \in \mathcal{F}$. Second, $\mu_c(\emptyset) = c_1\mu_1(\emptyset) + c_2\mu_2(\emptyset) = 0$ since μ_1 and μ_2 are measures. Third, if $\{F_i\}_{i\in\mathbb{N}} \in \mathcal{F}$ is a pairwise disjoint collection of sets,

$$\begin{aligned}\mu_c\left(\cup_{i\in\mathbb{N}}F_i\right) &= c_1\mu_1(\cup_{i\in\mathbb{N}}F_i)+c_2\mu_2(\cup_{i\in\mathbb{N}}F_i)\\ &= c_1\sum_{i\in\mathbb{N}}\mu_1(F_i)+c_2\sum_{i\in\mathbb{N}}\mu_2(F_i),\ \text{since } \mu_1 \text{ and } \mu_2 \text{ are measures}\\ &= \sum_{i\in\mathbb{N}}\left(c_1\mu_1(F_i)+c_2\mu_2(F_i)\right)=\sum_{i\in\mathbb{N}}\mu_c(F_i).\end{aligned}$$

b) (3 points) The verification that $\mu_\infty : \mathcal{F} \to [0,\infty]$ and $\mu_\infty(\emptyset) = 0$ follows the same arguments as in item a) when examining μ_c. For σ-additivity, note that if $\{F_j\}_{j\in\mathbb{N}} \in \mathcal{F}$ is a pairwise disjoint collection of sets,

$$\mu_\infty\left(\cup_{j\in\mathbb{N}}F_j\right)=\sum_{i=1}^{\infty}\alpha_i\mu_i\left(\cup_{j\in\mathbb{N}}F_j\right)=\sum_{i=1}^{\infty}\alpha_i\sum_{j=1}^{\infty}\mu_i\left(F_j\right)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\alpha_i\mu_i\left(F_j\right).$$

If we are able to interchange the sums in the last term, then we can write

$$\mu_\infty\left(\cup_{j\in\mathbb{N}}F_j\right)=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\alpha_i\mu_i\left(F_j\right)=\sum_{j=1}^{\infty}\mu_\infty\left(F_j\right),$$

completing the proof. Now, note that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\alpha_i\mu_i\left(F_j\right)=\lim_{n\to\infty}\lim_{m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\mu_i\left(F_j\right)=\sup_{n\in\mathbb{N}}\sup_{m\in\mathbb{N}}\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\mu_i\left(F_j\right)=\sup_{n\in\mathbb{N}}\sup_{m\in\mathbb{N}}S_{nm}$$

since the partial sums are increasing. Now, if $S_{nm} \in \mathbb{R}$, then

$$\sup_{n\in\mathbb{N}}\sup_{m\in\mathbb{N}}S_{nm}=\sup_{m\in\mathbb{N}}\sup_{n\in\mathbb{N}}S_{nm}.$$

Hence, to finish the proof, we require $\mu_i(F_j) < \infty$.

4. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. In this case, we call $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. Let $\nu := \mu|_{\mathcal{G}}$ be the restriction of μ to $\mathcal{G}$. That is, $\nu(G) = \mu(G)$ for all $G \in \mathcal{G}$. Is ν a measure? If μ is finite, is ν finite? If μ is a probability, is ν a probability?

   Answer: (2 points) Since $\emptyset \in \mathcal{G} \subset \mathcal{F}$, $\nu(\emptyset) = \mu(\emptyset) = 0$. If $\{A_i\}_{i\in\mathbb{N}} \in \mathcal{G}$ is a pairwise disjoint sequence, we have that $\{A_i\}_{i\in\mathbb{N}} \in \mathcal{F}$. Hence, $\nu(\cup_{i\in\mathbb{N}}A_i) = \mu(\cup_{i\in\mathbb{N}}A_i) = \sum_{i\in\mathbb{N}}\mu(A_i) = \sum_{i\in\mathbb{N}}\nu(A_i)$. Now, μ finite means that $\mu(\Omega) < \infty$. Since $\Omega \in \mathcal{G}$, $\nu(\Omega) = \mu(\Omega) < \infty$. The same holds for $\mu(\Omega) = 1$.

5. Show that a measure space $(\Omega, \mathcal{F}, \mu)$ is σ-finite if, and only if, there exists $\{F_n\}_{n\in\mathbb{N}} \in \mathcal{F}$ such that $\cup_{n\in\mathbb{N}}F_n = \Omega$ and $\mu(F_n) < \infty$ for all n.

   Answer: (2 points) ($\Rightarrow$) By definition, $(\Omega, \mathcal{F}, \mu)$ is σ-finite if there exists and increasing sequence $A_1 \subset A_2 \subset A_3 \cdots$ such that $\cup_{n\in\mathbb{N}}A_n = \Omega$ with $\mu(A_n) < \infty$ for all n. Hence, it suffices to let $F_n = A_n$. ($\Leftarrow$) Let $A_n = \cup_{j=1}^{n}F_j$. Then, $A_1 \subset A_2 \subset \cdots$ and $\cup_{n\in\mathbb{N}}A_n = \cup_{j\in\mathbb{N}}F_j = \Omega$. Also, $\mu(A_n) = \mu(\cup_{j=1}^{n}F_j) \leq \sum_{j=1}^{n}\mu(F_j) < \infty$ since the sum is finite and $\mu(F_j) < \infty$.