Econ 7818, Homework 1 - part 2, Professor Martins. Due date will be announced later, during class.

1. Let $\{E_j\}_{j \in J}$ be a collection of pairwise disjoint events. Show that if $P(E_j) > 0$ for each $j \in J$, then J is countable.

   Answer: (3 points) Let $C_n = \{E_j : P(E_j) > \frac{1}{n} \text{ and } j \in J\}$. By assumption the elements of C_n are disjoint events and

   $$P\left(\cup_{j_m} E_{j_m}\right) = \sum_{m=1}^{\infty} P(E_{j_m}) = \infty,$$

   where the last equality follows from the fact that $P(E_{j_m}) > 0$. So, it must be that C_n has finitely many elements. Also, $\{E_j\}_{j \in J} = \cup_{n=1}^{\infty} C_n$, which is countable since it is a countable union of finite sets.

2. Consider the extended real line, i.e., $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. Let $\bar{\mathcal{B}} := \mathcal{B}(\bar{\mathbb{R}})$ be defined as the collection of sets $\bar{B}$ such that $\bar{B} = B \cup S$ where $B \in \mathcal{B}(\mathbb{R})$ and $S \in \{\emptyset, \{-\infty\}, \{\infty\}, \{-\infty, \infty\}\}$. Show that $\bar{\mathcal{B}}$ is a σ-algebra and that it is generated by a collection of sets of the form $[a, \infty]$ where $a \in \mathbb{R}$.

   Answer: (2 points to show $\bar{\mathcal{B}}$ is a σ-algebra and 2 points for the rest) Let's first show that $\bar{\mathcal{B}}$ is a σ-algebra. Since $\bar{B} = B \cup S$ with $B \in \mathcal{B}(\mathbb{R})$, we can choose $B = \mathbb{R}$ and use $S = \{-\infty, \infty\}$ to conclude that $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} \in \bar{\mathcal{B}}$. Next, note that if $\bar{B} = B \cup S$ we have that $\bar{B}^c = B^c \cap S^c$. But the complement of a set S is an element of $\{\bar{\mathbb{R}}, \mathbb{R} \cup \{\infty\}, \mathbb{R} \cup \{-\infty\}, \mathbb{R}\}$. Hence, either 1) $\bar{B}^c = B^c \cap \bar{\mathbb{R}} = B^c \cup \emptyset \in \bar{\mathcal{B}}$ or, 2) $\bar{B}^c = B^c \cap (\mathbb{R} \cup \{\infty\}) = (B^c \cap \mathbb{R}) \cup \{\infty\}$ where $B^c \cap \mathbb{R} \in \mathcal{B}$ and consequently $\bar{B}^c \in \bar{\mathcal{B}}$ or, 3) $\bar{B}^c = B^c \cap (\mathbb{R} \cup \{-\infty\}) = (B^c \cap \mathbb{R}) \cup \{-\infty\}$ where $B^c \cap \mathbb{R} \in \mathcal{B}$ and consequently $\bar{B}^c \in \bar{\mathcal{B}}$ or, 4) $\bar{B}^c = B^c \cap \mathbb{R} \in \bar{\mathcal{B}}$.

   Lastly, letting $A_i = B_i \cup S$ for $B_i \in \mathcal{B}$ we have that $\cup_{i \in \mathbb{N}} A_i = \cup_{i \in \mathbb{N}} (B_i \cup S) = (\cup_{i \in \mathbb{N}} B_i) \cup S$. Since $\cup_{i \in \mathbb{N}} B_i \in \mathcal{B}$ we have that $\cup_{i \in \mathbb{N}} A_i \in \bar{\mathcal{B}}$.

   If $\bar{\mathcal{B}}$ is a σ-algebra and $\mathcal{C} = \{[a, \infty] : a \in \mathbb{R}\}$, we need to show that $\sigma(\mathcal{C}) = \bar{\mathcal{B}}$.

   First, note that $[a, \infty] = [a, \infty) \cup \{\infty\}$ and we know that $[a, \infty) \in \mathcal{B}$. Thus, $[a, \infty] \in \bar{\mathcal{B}}$ for all $a \in \mathbb{R}$. Then, $\sigma(\mathcal{C}) \subseteq \bar{\mathcal{B}}$.

   Second, observe that for $-\infty < a \leq b < \infty$ we have $[a, b) = [a, \infty] - [b, \infty] = [a, \infty] \cap [b, \infty]^c \in \sigma(\mathcal{C})$ since $\sigma(\mathcal{C})$ contains $[a, \infty]$ and $[b, \infty]^c$ by virtue of being a σ-algebra. Hence,

   $$\mathcal{B} \subseteq \sigma(\mathcal{C}) \subseteq \bar{\mathcal{B}}.$$

   Now,

   $$\{\infty\} = \cap_{i \in \mathbb{N}} [i, \infty], \ \{-\infty\} = \cap_{i \in \mathbb{N}} [-\infty, -i) = \cap_{i \in \mathbb{N}} [-i, \infty]^c$$

   which allows us to conclude that $\{\infty\}, \{-\infty\} \in \sigma(\mathcal{C})$. Hence, if $B \in \mathcal{B}$ all sets of the form

   $$B, B \cup \{\infty\}, B \cup \{-\infty\}, B \cup \{\infty\} \cup \{-\infty\}$$

   are in $\sigma(\mathcal{C})$. Hence, $\bar{\mathcal{B}} \subseteq \sigma(\mathcal{C})$. Combining this set. containment with $\sigma(\mathcal{C}) \subseteq \bar{\mathcal{B}}$ gives the result.

3. Let $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ be a random variable. Show that $M = \max\{X, 0\}$ and $m = \min\{0, -X\}$ are random variables.

   Answer: (2 points) $S_a^M = \{\omega : \max\{X, 0\} > a\}$. If $a < 0$, $S_a^M = \Omega$ and $S_a^M \in \mathcal{F}$. If $a \geq 0$, $S_a^M = \{\omega : X(\omega) > a\} \in \mathcal{F}$ by measurability of X.

   $S_a^m = \{\omega : \min\{0, -X\} > a\}$. If $a > 0$, $S_a^m = \emptyset$ and $S_a^m \in \mathcal{F}$. If $a \leq 0$, $S_a^m = \{\omega : X(\omega) < -a\} \in \mathcal{F}$ by measurability of X.

4. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $f : \Omega \to \mathbb{R}$ be a function. If $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ is a random variable such that $P(\{\omega : X(\omega) \neq f(\omega)\}) = 0$, then f is measurable.

   Answer: (3 points) We need to show that for any $B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{F}$. If $\mathcal{C} = \{(-\infty, a] : a \in \mathbb{R}\}$, then $\sigma(\mathcal{C}) = \mathcal{B}$ and it suffices to show that

   $$f^{-1}((-\infty, a]) = \{\omega : f(\omega) \leq a\} := S_a^f \in \mathcal{F}.$$

   Let $A = \{\omega : X(\omega) \neq f(\omega)\}$. Since, X is a random variable

   $$X^{-1}((-\infty, a]) = \{\omega : X(\omega) \leq a\} := S_a^X \in \mathcal{F}.$$

   Now, $S_a^f = (S_a^f \cap A) \cup (S_a^f \cap A^c)$. Note that $S_a^f \cap A^c = S_a^X \cap A^c \in \mathcal{F}$ by measurability of X and the fact that A is measurable (and so is A^c). Also, $S_a^f \cap A \subseteq A$ where A has measure zero. Hence, if $(\Omega, \mathcal{F}, P)$ is complete, such that all subsets of sets of measure zero are measurable, $S_a^f \cap A \in \mathcal{F}$. Hence, $S_a^f \in \mathcal{F}$.

5. Prove Theorem 1.6 in your notes with c) substituted by c') on Remark 1.4.

   Answer: (2 points) Note that if $A_1, A_2, \cdots \in \mathcal{F}$ we have that $A_1^c, A_2^c, \cdots \in \mathcal{F}$. Furthermore, since $A_1 \supseteq A_2 \supseteq \ldots$ we have that $A_1^c \subseteq A_2^c \subseteq \cdots$. Since, $\cap A_j = A$ we have that $\cup A_j^c = A^c$ and $A^c \in \mathcal{F}$. Hence, letting $B_1 = A_1^c$ and $B_j = A_j^c - A_{j-1}^c$ for $j = 2, 3, \ldots$, the proof follows as in Theorem 1.6.

6. If $E_1, E_2, \cdots, E_n$ are independent events, show that the probability that none of them occur is less than or equal to $\exp\left(-\sum_{i=1}^n P(E_i)\right)$

   Answer: (3 points). Let $f(x) = \exp(-x)$ and note that for $\lambda \in (0, 1)$, by Taylor's Theorem

   $$\exp(-x) = f(x) = f(0) + f^{(1)}(0)x + \frac{1}{2}f^{(2)}(\lambda x)x^2 = 1 - x + \frac{1}{2}\exp(-\lambda x)x^2$$

   Consequently, $1 - x \leq \exp(-x)$. Now, we are interested in the event $E = (\cup_{i=1}^n E_i)^c = \cap_{i=1}^n E_i^c$. But since the $E_1, E_2, \cdots, E_n$ are independent, so is the collection $E_1^c, E_2^c, \cdots, E_n^c$. Hence, $P(E) = \prod_{i=1}^n P(E_i^c) = \prod_{i=1}^n (1 - P(E_i)) \leq \prod_{i=1}^n \exp(-P(E_i)) = \exp\left(-\sum_{i=1}^n P(E_i)\right)$.

7. Let $\{A_n\}_{n \in \mathbb{N}}$ and $\{B_n\}_{n \in \mathbb{N}}$ be events (measurable sets) in a probability space with measure P with $\lim A_n = A$, $\lim B_n = B$, $P(B_n), P(B) > 0$ for all n. Show that $P(A_n|B) \to P(A|B)$, $P(A|B_n) \to P(A|B)$, $P(A_n|B_n) \to P(A|B)$ as $n \to \infty$.

   Answer: (3 points) Since $P(\cdot|B)$ is a probability measure (proved in the class notes), we have by continuity of probability measures that $P(A_n|B) \to P(A|B)$ if $\lim B_n = B$.

   Now, since $\lim B_n = B$ we have that $A \cap B_n \to A \cap B$. To see this, note that if $A \cap B_n := C_n$ then $D_j = \cup_{n=j}^\infty C_n = A \cap (\cup_{n=1}^\infty B_n)$. Then, $\limsup C_n = \cap_{j=1}^\infty D_j = \cap_{j=1}^\infty (A \cap \cup_{n=1}^\infty B_n) = A \cap B$. Defining $\liminf$ for C_n we can in similar fashion that $\liminf C_n = A \cap B$. Hence, by continuity of probability measures $P(A \cap B_n) \to P(A \cap B)$ and $P(B_n) \to P(B)$. Consequently,

   $$P(A|B_n) = \frac{P(A \cap B_n)}{P(B_n)} \to \frac{P(A \cap B)}{P(B)} = P(A|B).$$

   Lastly, since $A_n \cap B_n \to A \cup B$, using the same arguments

   $$P(A_n|B_n) = \frac{P(A_n \cap B_n)}{P(B_n)} \to \frac{P(A \cap B)}{P(B)} = P(A|B).$$

8. Let $(\Omega, \mathcal{F}, P)$ be a probability space and E_n for $n = 1, 2, \cdots$ be sets in $\mathcal{F}$. Show that if $\sum_{n=1}^\infty P(E_n) < \infty$ then $P\left(\underset{n \to \infty}{\text{limsup}} E_n\right) = 0$.

Answer:(2 points)

$$\begin{aligned} P\left(\limsup_{n\to\infty} E_n\right) &= P\left(\lim_{n\to\infty} \cup_{j\geq n} E_j\right) \\ &= \lim_{n\to\infty} P\left(\cup_{j\geq n} E_j\right) \text{ by continuity} \\ &\leq \limsup_{n\to\infty} \sum_{j=n}^{\infty} P(E_j) \text{ by subadditivity and definition of limsup.} \end{aligned}$$

Since $\sum_{n=1}^{\infty} P(E_n) < \infty$ it must be that $\sum_{j=n}^{\infty} P(E_j) \to 0$ as $n \to 0$. Consequently, $P\left(\limsup_{n\to\infty} E_n\right) = 0$.