Econ 7818, Homework 1 - part 2, Professor Martins. Due date 9/22/2023. Your answers should be in my mailbox by 5:00 PM.

1. Let $\{E_j\}_{j\in J}$ be a collection of pairwise disjoint events. Show that if $P(E_j) > 0$ for each $j \in J$, then J is countable.

   Answer: (3 points) Let $C_n = \{E_j : P(E_j) > \frac{1}{n} \text{ and } j \in J\}$. By assumption the elements of C_n are disjoint events and

   $$P\left(\cup_{j_m} E_{j_m}\right) = \sum_{m=1}^{\infty} P(E_{j_m}) = \infty,$$

   where the last equality follows from the fact that $P(E_{j_m}) > 0$. So, it must be that C_n has finitely many elements. Also, $\{E_j\}_{j\in J} = \cup_{n=1}^{\infty} C_n$, which is countable since it is a countable union of finite sets.

2. Consider the extended real line, i.e., $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. Let $\bar{\mathcal{B}} := \mathcal{B}(\bar{\mathbb{R}})$ be defined as the collection of sets $\bar{B}$ such that $\bar{B} = B \cup S$ where $B \in \mathcal{B}(\mathbb{R})$ and $S \in \{\emptyset, \{-\infty\}, \{\infty\}, \{-\infty, \infty\}\}$. Show that $\bar{\mathcal{B}}$ is a σ-algebra and that it is generated by a collection of sets of the form $[a, \infty]$ where $a \in \mathbb{R}$.

   Answer: (2 points to show $\bar{\mathcal{B}}$ is a σ-algebra and 2 points for the rest) Let's first show that $\bar{\mathcal{B}}$ is a σ-algebra. Since $\bar{B} = B \cup S$ with $B \in \mathcal{B}(\mathbb{R})$, we can choose $B = \mathbb{R}$ and use $S = \{-\infty, \infty\}$ to conclude that $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} \in \bar{\mathcal{B}}$. Next, note that if $\bar{B} = B \cup S$ we have that $\bar{B}^c = B^c \cap S^c$. But the complement of a set S is an element of $\{\bar{\mathbb{R}}, \mathbb{R} \cup \{\infty\}, \mathbb{R} \cup \{-\infty\}, \mathbb{R}\}$. Hence, either 1) $\bar{B}^c = B^c \cap \bar{\mathbb{R}} = B^c \cup \emptyset \in \bar{\mathcal{B}}$ or, 2) $\bar{B}^c = B^c \cap (\mathbb{R} \cup \{\infty\}) = (B^c \cap \mathbb{R}) \cup \{\infty\}$ where $B^c \cap \mathbb{R} \in \mathcal{B}$ and consequently $\bar{B}^c \in \bar{\mathcal{B}}$ or, 3) $\bar{B}^c = B^c \cap (\mathbb{R} \cup \{-\infty\}) = (B^c \cap \mathbb{R}) \cup \{-\infty\}$ where $B^c \cap \mathbb{R} \in \mathcal{B}$ and consequently $\bar{B}^c \in \bar{\mathcal{B}}$ or, 4) $\bar{B}^c = B^c \cap \mathbb{R} \in \bar{\mathcal{B}}$.

   Lastly, letting $A_i = B_i \cup S$ for $B_i \in \mathcal{B}$ we have that $\cup_{i\in\mathbb{N}} A_i = \cup_{i\in\mathbb{N}}(B_i \cup S) = (\cup_{i\in\mathbb{N}} B_i) \cup S$. Since $\cup_{i\in\mathbb{N}} B_i \in \mathcal{B}$ we have that $\cup_{i\in\mathbb{N}} A_i \in \bar{\mathcal{B}}$.

   If $\bar{\mathcal{B}}$ is a σ-algebra and $\mathcal{C} = \{[a, \infty] : a \in \mathbb{R}\}$, we need to show that $\sigma(\mathcal{C}) = \bar{\mathcal{B}}$.

   First, note that $[a, \infty] = [a, \infty) \cup \{\infty\}$ and we know that $[a, \infty) \in \mathcal{B}$. Thus, $[a, \infty] \in \bar{\mathcal{B}}$ for all $a \in \mathbb{R}$. Then, $\sigma(\mathcal{C}) \subseteq \bar{\mathcal{B}}$.

   Second, observe that for $-\infty < a \leq b < \infty$ we have $[a, b) = [a, \infty] - [b, \infty] = [a, \infty] \cap [b, \infty]^c \in \sigma(\mathcal{C})$ since $\sigma(\mathcal{C})$ contains $[a, \infty]$ and $[b, \infty]^c$ by virtue of being a σ-algebra. Hence,

   $$\mathcal{B} \subseteq \sigma(\mathcal{C}) \subseteq \bar{\mathcal{B}}.$$

   Now,

   $$\{\infty\} = \cap_{i\in\mathbb{N}}[i, \infty],\ \{-\infty\} = \cap_{i\in\mathbb{N}}[-\infty, -i) = \cap_{i\in\mathbb{N}}[-i, \infty]^c$$

   which allows us to conclude that $\{\infty\}, \{-\infty\} \in \sigma(\mathcal{C})$. Hence, if $B \in \mathcal{B}$ all sets of the form

   $$B, B \cup \{\infty\}, B \cup \{-\infty\}, B \cup \{\infty\} \cup \{-\infty\}$$

   are in $\sigma(\mathcal{C})$. Hence, $\bar{\mathcal{B}} \subseteq \sigma(\mathcal{C})$. Combining this set. containment with $\sigma(\mathcal{C}) \subseteq \bar{\mathcal{B}}$ gives the result.

3. If $E_1, E_2, \cdots, E_n$ are independent events, show that the probability that none of them occur is less than or equal to $\exp\left(-\sum_{i=1}^{n} P(E_i)\right)$.

   Answer: (3 points). Let $f(x) = \exp(-x)$ and note that for $\lambda \in (0, 1)$, by Taylor's Theorem

   $$\exp(-x) = f(x) = f(0) + f^{(1)}(0)x + \frac{1}{2}f^{(2)}(\lambda x)x^2 = 1 - x + \frac{1}{2}\exp(-\lambda x)x^2$$

Consequently, $1 - x \leq \exp(-x)$. Now, we are interested in the event $E = (\cup_{i=1}^n E_i)^c = \cap_{i=1}^n E_i^c$. But since the $E_1, E_2, \cdots, E_n$ are independent, so is the collection $E_1^c, E_2^c, \cdots, E_n^c$. Hence, $P(E) = \prod_{i=1}^n P(E_i^c) = \prod_{i=1}^n (1 - P(E_i)) \leq \prod_{i=1}^n \exp(-P(E_i)) = \exp\left(-\sum_{i=1}^n P(E_i)\right)$.

4. Let $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ be events (measurable sets) in a probability space with measure P with $\lim A_n = A$, $\lim B_n = B$, $P(B_n), P(B) > 0$ for all n. Show that $P(A_n|B) \to P(A|B)$, $P(A|B_n) \to P(A|B)$, $P(A_n|B_n) \to P(A|B)$ as $n \to \infty$.

   Answer: (3 points) Since $P(\cdot|B)$ is a probability measure (proved in the class notes), we have by continuity of probability measures that $P(A_n|B) \to P(A|B)$ if $\lim B_n = B$.

   Now, since $\lim B_n = B$ we have that $A \cap B_n \to A \cap B$. To see this, note that if $A \cap B_n := C_n$ then $D_j = \cup_{n=j}^\infty C_n = A \cap (\cup_{n=1}^\infty B_n)$. Then, $\limsup C_n = \cap_{j=1}^\infty D_j = \cap_{j=1}^\infty (A \cap \cup_{n=1}^\infty B_n) = A \cap B$. Defining $\liminf$ for C_n we can in similar fashion that $\liminf C_n = A \cap B$. Hence, by continuity of probability measures $P(A \cap B_n) \to P(A \cap B)$ and $P(B_n) \to P(B)$. Consequently,

$$P(A|B_n) = \frac{P(A \cap B_n)}{P(B_n)} \to \frac{P(A \cap B)}{P(B)} = P(A|B).$$

   Lastly, since $A_n \cap B_n \to A \cup B$, using the same arguments

$$P(A_n|B_n) = \frac{P(A_n \cap B_n)}{P(B_n)} \to \frac{P(A \cap B)}{P(B)} = P(A|B).$$

5. Let $(\Omega, \mathcal{F}, P)$ be a probability space and E_n for $n = 1, 2, \cdots$ be sets in $\mathcal{F}$. Show that if $\sum_{n=1}^\infty P(E_n) < \infty$ then $P\left(\underset{n\to\infty}{\text{limsup}} E_n\right) = 0$.

   Answer:(2 points)

$$\begin{aligned} P\left(\underset{n\to\infty}{\text{limsup}} E_n\right) &= P\left(\lim_{n\to\infty} \cup_{j\geq n} E_j\right) \\ &= \lim_{n\to\infty} P\left(\cup_{j\geq n} E_j\right) \text{ by continuity} \\ &\leq \underset{n\to\infty}{\text{limsup}} \sum_{j=n}^\infty P(E_j) \text{ by subadditivity and definition of limsup.} \end{aligned}$$

   Since $\sum_{n=1}^\infty P(E_n) < \infty$ it must be that $\sum_{j=n}^\infty P(E_j) \to 0$ as $n \to 0$. Consequently, $P\left(\underset{n\to\infty}{\text{limsup}} E_n\right) = 0$.

6. Let $(\mathbb{X}, \bar{\mathcal{F}}, \bar{\mu})$ be the measure space defined in Theorem 1.15 of your notes and $\mathcal{C} = \{G \in \mathbb{X} : \exists\, A,\, B \in \mathcal{F} \ni A \subset G \subset B \text{ and } \mu(B - A) = 0\}$. Show that $\bar{\mathcal{F}} = \mathcal{C}$.

   Answer: (3 points) $G \in \bar{\mathcal{F}} \implies G = A \cup M$ where $A \in \mathcal{F}$ and $M \in \mathcal{S}$. $M \in \mathcal{S} \implies \exists N \in \mathcal{N}_\mu \ni M \subset N$. Then,

$$A \subset G = A \cup M \subset A \cup N := B \in \mathcal{F}.$$

   Now, $\mu(B - A) = \mu(B \cup A^c) = \mu((A \cup N) - A) \leq \mu(N) = 0$. Thus, $G \in \mathcal{C}$.

   $G \in \mathcal{C} \implies \exists A, B \in \mathcal{F} \ni A \subset G \subset B$ and $\mu(B - A) = 0$. Since $A \subset G \subset B$ we have that $G - A \subset B - A$, and since $B - A$ is a μ-null set $G - A \in \mathcal{S}$. Now, $G = A \cup (G - A)$, and since $A \in \mathcal{F}$, $G \in \bar{\mathcal{F}}$.

7. Let μ be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\mu([-n,n)) < \infty$ for all $n \in \mathbb{N}$. Define,

$$F_\mu(x) := \begin{cases} \mu([0,x)) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\mu([x,0)) & \text{if } x < 0. \end{cases}$$

Show that $F_\mu : \mathbb{R} \to \mathbb{R}$ is monotonically increasing and left continuous.

Answer: (4 points) Given that $\mu([-n,n)) < \infty$, F_μ takes values in $\mathbb{R}$. First, we show that all $x < x'$, $F_\mu(x) \leq F_\mu(x')$. There are three cases to be considered

(a) $(0 \leq x < x')$: if $0 < x < x'$, $F_\mu(x') - F_\mu(x) = \mu([0,x')) - \mu([0,x))$. Since $[0,x') = [0,x) \cup [x,x')$, σ-additivity of μ gives $\mu([0,x')) = \mu([0,x)) + \mu([x,x'))$ or $\mu([x,x')) = \mu([0,x')) - \mu([0,x)) = F_\mu(x') - F_\mu(x) \geq 0$. If $x = 0$, $F_\mu(x') - F_\mu(0) = \mu([0,x')) \geq 0$.

(b) $(x < 0 \leq x')$: If $x' > 0$, $F_\mu(x') - F_\mu(x) = \mu([0,x')) + \mu([x,0)) \geq 0$. If $x' = 0$, $F_\mu(0) - F_\mu(x) = \mu([x,0)) \geq 0$.

(c) $(x < x' < 0)$: $F_\mu(x') - F_\mu(x) = -\mu([x',0)) + \mu([x,0))$. Since $[x,0) = [x,x') \cup [x',0)$, σ-additivity of μ gives $\mu([x,0)) = \mu([x,x')) + \mu([x',0))$ or $\mu([x,0)) - \mu([x',0)) = F_\mu(x') - F_\mu(x) = \mu([x,x')) \geq 0$.

Second, we must show that $\lim_{n\to\infty} F_\mu(x - h_n) = F_\mu(x)$ for all $x \in \mathbb{R}$. Let $n \in \mathbb{N}$, $h_1 \geq h_2 \geq h_3 \geq \cdots$ with $h_n \downarrow 0$ as $n \to \infty$, and $h_1 > 0$. There are three cases to consider.

(a) $(x > 0)$: Choose $h_1 \in (0,x)$ and define $A_n = [0, x - h_n)$. Then, $A_1 \subset A_2 \subset \cdots$ and $\lim_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n = [0,x)$. By continuity of measure from below,

$$\lim_{n\to\infty} F_\mu(x - h_n) = \lim_{n\to\infty} \mu([0, x - h_n)) = \mu([0,x)) = F_\mu(x).$$

(b) $(x = 0)$: Define $A_n = [-h_n, 0)$. Then, $A_1 \supset A_2 \supset \cdots$ and $\lim_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} A_n = \emptyset$. By continuity of measures from above, and given that $\mu([-h_1, 0)) < \infty$,

$$\lim_{n\to\infty} F_\mu(-h_n) = \lim_{n\to\infty} \mu([-h_n, 0)) = \mu(\emptyset) = 0 = F_\mu(0).$$

(c) $(x < 0)$: Define $A_n = [x - h_n, 0)$. Then, $A_1 \supset A_2 \supset \cdots$ and $\lim_{n\to\infty} A_n = \cap_{n=1}^\infty A_n = [x,0)$. By continuity of measures from above and given that $\mu([x - h_1, 0)) < \infty$,

$$\lim_{n\to\infty} F_\mu(x - h_n) = \lim_{n\to\infty} -\mu([x - h_n, 0)) = -\mu([x,0)) = F_\mu(x).$$