Econ 7818 - Fall 2022, Homework 2 - part 1, Professor Martins. Due date: TBA

1. Let $(\mathbb{X}_i, \mathcal{F}_i)$ for $i = 1, 2, 3$ be measurable spaces and $f : \mathbb{X}_1 \to \mathbb{X}_2$, $g : \mathbb{X}_2 \to \mathbb{X}_3$ be $\mathcal{F}_1 - \mathcal{F}_2$ and $\mathcal{F}_2 - \mathcal{F}_3$ measurable. Show that $g \circ f : \mathbb{X}_1 \to \mathbb{X}_3$ is $\mathcal{F}_1 - \mathcal{F}_3$ measurable.

   Answer: See Theorem 4.5.

2. Suppose $(\Omega, \mathcal{F})$ and $(\mathbb{Y}, \mathcal{G})$ are measure spaces and $f : \Omega \to \mathbb{Y}$. Show that: a) $I_{f^{-1}(A)}(\omega) = (I_A \circ f)(\omega)$ for all ω; b) f is measurable if, and only if, $\sigma(\{f^{-1}(A) : A \in \mathcal{G}\}) \subset \mathcal{F}$.

   Answer: a) For any subset $A \subseteq Y$, we have $f^{-1}(A) = \{\omega : f(\omega) \in A\}$. Then,

$$I_{f^{-1}(A)}(\omega) = I_{\{\omega: f(\omega) \in A\}}(\omega) = I_A(f(\omega)) = (I_A \circ f)(\omega).$$

   b) Since f is measurable, $f^{-1}(\mathcal{G}) \subseteq \mathcal{F}$. By monotonicity of σ-algebras, $\sigma(f^{-1}(\mathcal{G})) = \sigma(\{f^{-1}(A) : A \in \mathcal{G}\}) \subseteq \mathcal{F}$. Now, $\sigma(f^{-1}(\mathcal{G})) = f^{-1}(\sigma(\mathcal{G})) = f^{-1}(\mathcal{G}) \subset \mathcal{F}$. The last set containment implies measurability.

3. Show that for any function $f : \mathbb{X} \to \mathbb{Y}$ and any collection of subsets $\mathcal{G}$ of $\mathbb{Y}$, $f^{-1}(\sigma(\mathcal{G})) = \sigma(f^{-1}(\mathcal{G}))$

   Answer: See Theorem 4.1.

4. For $n \in \mathbb{N}$, let $P_1, P_2, \cdots, P_n$ be measures on $\mathcal{F}$ and $a_1, a_2, \cdots, a_n$ be non-negative real numbers. Show that

$$m(E) = \sum_{j=1}^{n} a_j P_j(E) \text{ for } E \in \mathcal{F}$$

   is a measure on $\mathcal{F}$. That is, linear combination of measures are measures.

   Answer: (2 points) We verify the properties of a measure. 1. $\lambda(E) = \sum_{j=1}^{n} a_j P_j(E)$, and since $a_j \geq 0$ and $P_j \geq 0$ we have that $\lambda(E) \geq 0$; 2. If $E = \emptyset$ then $\lambda(\emptyset) = \sum_{j=1}^{n} a_j P_j(\emptyset) = 0$ since $P_j(\emptyset) = 0$; 3. Let $\{E_i\}_{i \in \mathbb{N}}$ be a disjoint collection and $E = \cup_{i \in \mathbb{N}} E_i$. Then,

$$\begin{aligned} \lambda(E) &= \sum_{j=1}^{n} a_j P_j(E) = \sum_{j=1}^{n} a_j P_j(\cup_{i \in \mathbb{N}} E_i) \\ &= \sum_{j=1}^{n} a_j \sum_{i=1}^{\infty} P_j(E_i) = \sum_{i=1}^{\infty} \sum_{j=1}^{n} a_j P_j(E_i) = \sum_{i=1}^{\infty} \lambda(E_i) \end{aligned}$$

5. Prove Remark 3.3 on your notes.

   Answer: (2 points) Since f is simple and $f \leq f$, f is one of the simple functions (denoted by ϕ) appearing in Definition 3.3 of the class notes. Hence, $\int f d\mu \geq I_\mu(f)$. Also, if ϕ is a simple function such that $\phi \leq f$, by monotonicity of the integral of simple functions we have $I_\mu(\phi) \leq I_\mu(f)$, hence

$$\sup_{\phi} I_\mu(\phi) := \int f d\mu \leq I_\mu(f).$$

   Combining the two inequalities we have $\int f d\mu = I_\mu(f)$.

6. Let $(\mathbb{X}, \mathcal{F})$ be a measurable space and $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of measures defined on it. Noting that $\mu = \sum_{n\in\mathbb{N}} \mu_n$ is also a measure on $(\mathbb{X}, \mathcal{F})$ (you don't have to prove this), show that

$$\int_{\mathbb{X}} f d\mu = \sum_{n\in\mathbb{N}} \int_{\mathbb{X}} f d\mu_n$$

for f non-negative and measurable.

Answer: (6 points) First, let $f = I_F \geq 0$ for $F \in \mathcal{F}$. Then, f is measurable and

$$\int_{\mathbb{X}} f d\mu = \int_{\mathbb{X}} I_F d\mu = \mu(F) = \sum_{n\in\mathbb{N}} \mu_n(F) = \sum_{n\in\mathbb{N}} \int_{\mathbb{X}} I_F d\mu_n = \sum_{n\in\mathbb{N}} \int_{\mathbb{X}} f d\mu_n.$$

Hence, the result holds for indicator functions. Now, consider a simple non-negative function $f = \sum_{j=0}^{m} a_j I_{A_j}$ where $a_j \geq 0$ and $A_j \in \mathcal{F}$. Then,

$$\begin{aligned}\int_{\mathbb{X}} f d\mu &= \int_{\mathbb{X}} \sum_{j=0}^{m} a_j I_{A_j} d\mu = \sum_{j=0}^{m} a_j \int_{\mathbb{X}} I_{A_j} d\mu = \sum_{j=0}^{m} a_j \mu(A_j) = \sum_{j=0}^{m} a_j \sum_{n\in\mathbb{N}} \mu_n(A_j) \\ &= \sum_{n\in\mathbb{N}} \sum_{j=0}^{m} a_j \mu_n(A_j) = \sum_{n\in\mathbb{N}} \int_{\mathbb{X}} f d\mu_n.\end{aligned}$$

Hence, the result holds for simple non-negative functions. Lastly, let f be non-negative and measurable. By Theorem 3.3 in the class notes, there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ of non-negative, non-decreasing, measurable simple function such that $\sup\limits_{n\in\mathbb{N}} \phi_n = f$. By Beppo-Levi's Theorem

$$\int_{\mathbb{X}} f d\mu = \sup_{n\in\mathbb{N}} \int_{\mathbb{X}} \phi_n d\mu.$$

Hence,

$$\begin{aligned}\int_{\mathbb{X}} f d\mu &= \sup_{n\in\mathbb{N}} \int_{\mathbb{X}} \phi_n d\mu = \sup_{n\in\mathbb{N}} \sum_{j=1}^{\infty} \int_{\mathbb{X}} \phi_n d\mu_j \\ &= \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} \sum_{j=1}^{m} \int_{\mathbb{X}} \phi_n d\mu_j \text{ since } \textstyle\int_{\mathbb{X}} \phi_n d\mu_j \text{ is nondecreasing.} \\ &= \sup_{m\in\mathbb{N}} \sup_{n\in\mathbb{N}} \sum_{j=1}^{m} \int_{\mathbb{X}} \phi_n d\mu_j = \sup_{m\in\mathbb{N}} \lim_{n\to\infty} \sum_{j=1}^{m} \int_{\mathbb{X}} \phi_n d\mu_j \\ &= \sup_{m\in\mathbb{N}} \sum_{j=1}^{m} \lim_{n\to\infty} \int_{\mathbb{X}} \phi_n d\mu_j \\ &= \sup_{m\in\mathbb{N}} \sum_{j=1}^{m} \int_{\mathbb{X}} \lim_{n\to\infty} \phi_n d\mu_j \text{ by Beppo-Levi's Theorem} \\ &= \sup_{m\in\mathbb{N}} \sum_{j=1}^{m} \int_{\mathbb{X}} f d\mu_j = \sum_{j\in\mathbb{N}} \int_{\mathbb{X}} f d\mu_j.\end{aligned}$$

7. Prove Theorems 3.1 on your notes.

Answer: (2 points for proving Theorem 3.1 and 4 points for proving Theorem 3.8). The proof of Theorem 3.1 is direct from the definition of simple functions. For Theorem 3.8, just consider the positive and negative parts of the functions in 1 through 4 and show that their integrals are finite or use Theorems 3.7 in combination with Theorem 3.5.