Econ 7818 - Fall 2023, Homework 2 - part 1, Professor Martins. Due date: TBA

1. Suppose $(\Omega, \mathcal{F})$ and $(\mathbb{Y}, \mathcal{G})$ are measure spaces and $f : \Omega \to \mathbb{Y}$. Show that: a) $I_{f^{-1}(A)}(\omega) = (I_A \circ f)(\omega)$ for all ω; b) f is measurable if, and only if, $\sigma(\{f^{-1}(A) : A \in \mathcal{G}\}) \subset \mathcal{F}$.

   Answer: a) For any subset $A \subset Y$, we have $f^{-1}(A) = \{\omega : f(\omega) \in A\}$. Then,

   $$I_{f^{-1}(A)}(\omega) = I_{\{\omega : f(\omega) \in A\}}(\omega) = I_A(f(\omega)) = (I_A \circ f)(\omega).$$

   b) Since f is measurable, $f^{-1}(\mathcal{G}) \subset \mathcal{F}$. By monotonicity of σ-algebras, $\sigma(f^{-1}(\mathcal{G})) = \sigma(\{f^{-1}(A) : A \in \mathcal{G}\}) \subset \mathcal{F}$. Now, $\sigma(f^{-1}(\mathcal{G})) = f^{-1}(\sigma(\mathcal{G})) = f^{-1}(\mathcal{G}) \subset \mathcal{F}$. The last set containment implies measurability.

2. Show that for any function $f : \mathbb{X} \to \mathbb{Y}$ and any collection of subsets $\mathcal{G}$ of $\mathbb{Y}$, $f^{-1}(\sigma(\mathcal{G})) = \sigma(f^{-1}(\mathcal{G}))$

   Answer: $f^{-1}(\sigma(\mathcal{G}))$ is a σ-algebra associated with $\mathbb{X}$. Since $\mathcal{G} \subset \sigma(\mathcal{G})$, $f^{-1}(\mathcal{G}) \subset f^{-1}(\sigma(\mathcal{G}))$ and consequently $\sigma(f^{-1}(\mathcal{G})) \subset f^{-1}(\sigma(\mathcal{G}))$.

   Now, as in Theorem 3.1, $\mathcal{U} = \{U \in 2^{\mathbb{Y}} : f^{-1}(U) \in \sigma(f^{-1}(\mathcal{G}))\}$ is a σ-algebra. By definition of $\mathcal{U}$

   $$f^{-1}(\mathcal{U}) \subset \sigma(f^{-1}(\mathcal{G})).$$

   Also, $\mathcal{G} \subset \mathcal{U}$ since $f^{-1}(\mathcal{G}) \subset f^{-1}(\mathcal{U}) \subset \sigma(f^{-1}(\mathcal{G}))$. Since $\mathcal{U}$ is a σ-algebra we have that $\sigma(\mathcal{G}) \subset \mathcal{U}$. So,

   $$f^{-1}(\sigma(\mathcal{G})) \subset f^{-1}(\mathcal{U}) \subset \sigma(f^{-1}(\mathcal{C})).$$

   The last set containment combined with the reverse obtained on the last paragraph completes the proof.

3. Let $i \in I$ where I is an arbitrary index set. Consider $f_i : (\mathbb{X}, \mathcal{F}) \to (\mathbb{X}_i, \mathcal{F}_i)$.

   (a) Show that for all i, the smallest σ-algebra associated with $\mathbb{X}$ that makes f_i measurable is given by $f_i^{-1}(\mathcal{F}_i)$.

   (b) Show that $\sigma\left(\bigcup\limits_{i \in I} f_i^{-1}(\mathcal{F}_i)\right)$ is the smallest σ-algebra associated with $\mathbb{X}$ that makes all f_i simultaneously measurable.

   Answer: a) f_i is measurable if $f_i^{-1}(\mathcal{F}_i) \subset \mathcal{F}$. But by monotonicity of $\sigma(\cdot)$ we have $\sigma(f_i^{-1}(\mathcal{F}_i)) = f_i^{-1}(\mathcal{F}_i) \subset \mathcal{F}$ since $f_i^{-1}(\mathcal{F}_i)$ is a σ-algebra. b) $f_i^{-1}(\mathcal{F}_i) \subset \mathcal{F}$ for all $i \in I$ because f_i is measurable. But any sub-σ-algebra of $\mathcal{F}$ that makes all f_i measurable functions must contain all $f_i^{-1}(\mathcal{F}_i)$, i.e., $\bigcup\limits_{i \in I} f_i^{-1}(\mathcal{F}_i)$. However, unions of σ-algebras are not necessarily σ-algebras. Hence, we consider $\sigma\left(\bigcup\limits_{i \in I} f_i^{-1}(\mathcal{F}_i)\right)$, the smallest σ-algebra that makes all f_i simultaneously measurable.

4. Let F_μ be as in problem 7 in Homework 1 - part 2. Show that $v_{F_\mu}([a, b)) := F_\mu(b) - F_\mu(a)$ for all $a < b$, $a, b \in \mathbb{R}$ has a unique extension to a measure in $\mathcal{B}$ and conclude that $\mu = v_{F_\mu}$.

   Answer: Recall that $\mathcal{S} = \{[a, b) : a \leq b,\ a, b \in \mathbb{R}\}$ is a semi-ring (if $a = b$, $[a, a) = \emptyset$). Given F_μ, we define $\nu_{F_\mu} : \mathcal{S} \to [0, \infty)$ as $\nu_{F_\mu}([a, b)) = F_\mu(b) - F_\mu(a)$ for all $a \leq b$. Since F_μ is monotonically increasing, $F_\mu(b) - F_\mu(a) \geq 0$ and $\nu_{F_\mu}([a, a) = \emptyset) = F_\mu(a) - F_\mu(a) = 0$. Also, ν_{F_μ} is finitely additive since for $a < c < b$, we have that $[a, b) = [a, c) \cup [c, b)$ and $\nu_{F_\mu}([a, b)) = F_\mu(b) - F_\mu(a) = F_\mu(c) - F_\mu(a) + F_\mu(b) - F_\mu(c) = \nu_{F_\mu}([a, c)) + \nu_{F_\mu}([c, b))$. We now show that ν_{F_μ} is σ-additive, i.e., for $[a_n, b_n)$, $n \in \mathbb{N}$ a disjoint collection such that $[a, b) = \bigcup\limits_{n \in \mathbb{N}} [a_n, b_n)$, we have $\nu_{F_\mu}([a, b)) = \sum\limits_{n \in \mathbb{N}} \nu_{F_\mu}([a_n, b_n))$. Fix ϵ_n, $\epsilon > 0$

and note that $(a_n-\epsilon_n,b_n)\supset[a_n,b_n)$. Hence, $\underset{n\in\mathbb{N}}{\cup}(a_n-\epsilon_n,b_n)\supset\underset{n\in\mathbb{N}}{\cup}[a_n,b_n)=[a,b)\supset[a,b-\epsilon]$. Since $\underset{n\in\mathbb{N}}{\cup}(a_n-\epsilon_n,b_n)$ is an open cover for the compact set $[a,b-\epsilon]$, by the Heine-Borel Theorem, there exists $N\in\mathbb{N}$ such that

$$\cup_{n=1}^{N}[a_n-\epsilon_n,b_n)\supset\cup_{n=1}^{N}(a_n-\epsilon_n,b_n)\supset[a,b-\epsilon]\supset[a,b-\epsilon). \tag{1}$$

Now, since $\cup_{n\in\mathbb{N}}[a_n,b_n)=[a,b)$ we have $\cup_{n=1}^{N}[a_n,b_n)\subset[a,b)$ and

$$\nu_{F_\mu}([a,b))\geq\nu_{F_\mu}\left(\cup_{n=1}^{N}[a_n,b_n)\right)=\sum_{n=1}^{N}\nu_{F_\mu}\left([a_n,b_n)\right)\text{ by finite additivity.}$$

Hence, we have

$$\begin{aligned}
0&\leq\nu_{F_\mu}([a,b))-\sum_{n=1}^{N}\nu_{F_\mu}\left([a_n,b_n)\right)\\
&=\nu_{F_\mu}([a,b-\epsilon))+\nu_{F_\mu}([b-\epsilon,b))-\sum_{n=1}^{N}\left(\nu_{F_\mu}([a_n-\epsilon_n,b_n))-\nu_{F_\mu}([a_n-\epsilon_n,a_n))\right)\\
&=\nu_{F_\mu}([a,b-\epsilon))-\sum_{n=1}^{N}\nu_{F_\mu}([a_n-\epsilon_n,b_n))\text{ this term }<0\text{ by (1)}\\
&+\nu_{F_\mu}([b-\epsilon,b))+\sum_{n=1}^{N}\nu_{F_\mu}([a_n-\epsilon_n,a_n))\\
&\leq\nu_{F_\mu}([b-\epsilon,b))+\sum_{n=1}^{N}\nu_{F_\mu}([a_n-\epsilon_n,a_n))=F_\mu(b)-F_\mu(b-\epsilon)+\sum_{n=1}^{N}(F_\mu(a_n)-F_\mu(a_n-\epsilon_n)).
\end{aligned}$$

By left-continuity of F_μ, we can choose ϵ such that $F_\mu(b)-F_\mu(b-\epsilon)<\eta/2$ and ϵ_n such that $F_\mu(a_n)-F_\mu(a_n-\epsilon_n)<2^{-n}\,\eta/2$. Hence,

$$0\leq\nu_{F_\mu}([a,b))-\sum_{n=1}^{N}\nu_{F_\mu}\left([a_n,b_n)\right)\leq\frac{\eta}{2}\left(1+\sum_{n=1}^{N}2^{-n}\right).$$

Letting $N\to\infty$ we have that $\nu_{F_\mu}([a,b))=\sum_{n=1}^{\infty}\nu_{F_\mu}\left([a_n,b_n)\right)$.

Since ν_{F_μ} is a pre-measure on a semi-ring, by Carathéodory's Theorem, it has a unique extension to $\sigma(\mathcal{S})=\mathcal{B}(\mathbb{R})$. Furthermore, since for $n\in\mathbb{N}$, $[-n,n)\uparrow\mathbb{R}$ and $\nu_{F_\mu}([-n,n))=F_\mu(n)-F_\mu(-n)=\mu([0,n))+\mu([-n,0)))<\infty$, this extension is unique.

It suffices to verify that $\nu_{F_\mu}=\mu$ on $\mathcal{S}$, since ν_{F_μ} extends uniquely to $\mathcal{B}(\mathbb{R})$. In fact,

Case 1 ($0\leq a<b$): $\nu_{F_\mu}([a,b))=F_\mu(b)-F_\mu(a)=\mu([0,b))-\mu([0,a))=\mu([0,a))+\mu([a,b))-\mu([0,a))=\mu([a,b))$, since $[0,b)=[0,a)\cup[a,b)$.

Case 2 ($a<0<b$): $\nu_{F_\mu}([a,b))=F_\mu(b)-F_\mu(a)=\mu([0,b))+\mu([a,0))=\mu([a,b))$, since $[a,b)=[a,0)\cup[0,b)$.

Case 3 ($a<b\leq0$): $\nu_{F_\mu}([a,b))=F_\mu(b)-F_\mu(a)=-\mu([b,0))+\mu([a,0))=\mu([a,b))$, since $[a,b)=[a,0)-[b,0)$.

5. Consider the measurable space $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ and the one-dimensional Lebesgue measure λ^1. For any measure m on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ a point $x\in\mathbb{R}$ is called an atom if $m(\{x\})>0$. Show that every measure m with no atoms can be written as m in Theorem 3.4 in your notes, with $\mu=\lambda^1$. Hint: use question 4

in this homework.

Answer: Note that F_μ from question 4 is not right-continuous. Let $x \geq 0$, $h_n > 0$ and $A_n = [0, x+h_n)$. Then $A_1 \supseteq A_2 \supseteq \cdots$ and $\lim_{n\to\infty} A_n = \cap_{n=1}^{\infty} A_n = [0,x] = [0,x) \cup \{x\}$. Hence, $\lim_{n\to\infty} F_\mu(x+h_n) = \mu([0,x]) = F_\mu(x) + \mu(\{x\})$. Also, if $x < 0$, $0 < h_n < -x$ and $A_n = [x+h_n, 0)$. Then, $A_1 \subset A_2 \subset \cdots$ and $\lim_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n = (x,0) = [x,0) - \{x\}$. Hence, $\lim_{n\to\infty} F_\mu(x+h_n) = -\mu((x,0)) = F_\mu(x) + \mu(\{x\})$. Hence, unless $\mu(\{x\}) = 0$ we have $\lim_{n\to\infty} F_\mu(x+h_n) \neq F_\mu(x)$. In fact, for any $x \in \mathbb{R}$, a point of continuity of F_μ,

$$\begin{aligned}\mu(\{x\}) &= \mu\left(\bigcap_{n\in\mathbb{N}}\left[x, x+\frac{1}{n}\right)\right) = \lim_{n\to\infty}\mu\left(\left[x, x+\frac{1}{n}\right)\right) \\ &= \lim_{n\to\infty} F_\mu\left(x+\frac{1}{n}\right) - F_\mu(x) = 0 \text{ by right continuity of } F_\mu.\end{aligned}$$

Thus, F_μ is continuous at x if, and only if, $\mu(\{x\}) = 0$.

In the notation of this question, F_m is continuous if if, and only if, $m(\{x\}) = 0$, that is, there are no atoms. Now,

$$\begin{aligned}m([a,b)) &= F_m(b) - F_m(a) = \lambda^1([F_m(a), F_m(b))) \\ &= \lambda^1(F_m([a,b))), \text{ by continuity} \\ &= (\lambda^1 \circ F_m)([a,b)).\end{aligned}$$

The last equality follows from defining the inverse map f^{-1} in Theorem 3.4 as the inverse function of $F_m$when it exists, or as the generalized inverse F_m^- as in your notes.

6. Prove Remark 4.3 in your class notes.

   Answer: Since f is simple and $f \leq f$ it is an admissible function in the set over which the supremum is being taken. Hence, $I_\mu(f) \leq \int f d\mu$. If g is simple and $g \leq f$, $I_\mu(g) \leq I_\mu(f)$ and

$$\int f d\mu = \sup\{I_\mu(g) : g \leq f\} \leq I_\mu(f).$$