Econ 7818, Homework 2, part 2, Professor Martins.
Due date: Parts 1 and 2 of Homework 2 are due October 13 in class.

1. Use Markov's inequality in your notes to prove the following for $a > 0$ and $g : (0, \infty) \to (0, \infty)$ that is increasing:

$$P(|X(\omega)| \geq a) \leq \frac{1}{g(a)} \int g(|X|)dP$$

Answer: Since g is increasing, $\{\omega : |X(\omega)| \geq a\} = \{\omega : g(|X(\omega)|) \geq g(a)\}$. Hence, since g is positive

$$g(a)I_{\{\omega:|X(\omega)|\geq a\}} = g(a)I_{\{\omega:g(|X(\omega)|)\geq g(a)\}} \leq g(|X(\omega)|).$$

Integrating both sides we have $g(a)P(\{\omega : |X(\omega)| \geq a\}) \leq \int g(|X(\omega)|)dP$. This completes the proof as $g(a) > 0$.

2. Let X be a random variable defined in the probability space $(\Omega, \mathcal{F}, P)$ with $E(X^2) < \infty$. Consider a function $f : \mathbb{R} \to \mathbb{R}$. What restrictions are needed on f to guarantee that $f(X)$ is a random variable with $E(f(X)^2) < \infty$?

Answer: (3 points) Recall that if $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, we say that X is a random variable (measurable real valued function) if, and only if, for all $B \in \mathcal{B}_{\mathbb{R}}$ we have $X^{-1}(B) \in \mathcal{F}$. Hence, if $h(\omega) := f(X(\omega)) = (f \circ X)(\omega) : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ we require that for all $B \in \mathcal{B}_{\mathbb{R}}$ we have $h^{-1}(B) = (f \circ X)^{-1}(B) = X^{-1}(f^{-1}(B)) \in \mathcal{F}$. That is, $f^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$.

Since X is a random variable (measurable) and given that $f^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$ for all $B \in \mathcal{B}_{\mathbb{R}}$, $f(X)$ is a random variable (measurable). Since the f^2 is a continuous function of f, f^2 is also a random variable (measurable). Hence, we can consider the integrability (or not) of $f(X)^2$, i.e., whether or not $E(f(X)^2) < \infty$. We give two general restrictions on f that give $E(f(X)^2) < \infty$. First, suppose that $\sup_{\omega\in\Omega} |h(\omega)| = \sup_{\omega\in\Omega} |(f \circ X)(\omega)| < C$. Then,

$$\left|\int f^2 dP\right| \leq \int h^2 dP \leq C^2 \int dP = C^2.$$

Second, suppose that $h^2 \leq X^2$ for all $\omega \in \Omega$. Then, $\int h^2 dP \leq \int X^2 dP < \infty$.

Note that, in general, it is not true that $E(f(X)^2) < \infty$ even if $E(X^2) < \infty$. For example, suppose that $X \sim U[0, 1]$. Then, $E(X^2) = 1/3$. Now, let $Y := f(X) = \tan\left(\pi(X - \frac{1}{2})\right)$ and we can easily obtain that the probability density of Y is

$$f_Y(y) = \left|\frac{d}{dy}f^{-1}(y)\right| = \left|\frac{d}{dy}\left(\frac{1}{2} + \frac{1}{\pi}\arctan(y)\right)\right| = \frac{1}{\pi}\frac{1}{1+y^2},\ y \in \mathbb{R}.$$

But this is the Cauchy density and $\int y^2 f_Y(y)dy$ does not exist.

3. Let $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B})$ be a random variable. Show that if $V(X) := E\left((X - E(X))\right)^2 = 0$ then X is a constant with probability 1.

Answer: (2 points) From your notes, if $\int_\Omega X^2 dP = 0$ then $X^2 = 0$ almost everywhere. If N is a null set $\int_\Omega X^2 dP = \int_N X^2 dP + \int_{N^c} X^2 dP = \int_N X^2 dP + \int_{N^c} 0 dP = 0$. Thus, $P(X^2 = x) = 0$ for $x \neq 0$ and $P(X^2 = 0) = 1$. But this is equivalent to $P(X = 0) = 1$. Hence, $V(X) = E\left((X - E(X))\right)^2 = 0$ implies $P(X - E(X) = 0) = P(X = E(X)) = 1$.

4. Show that the distribution F_X associated with the random variable X is continuous at x if, and only if, $P(X = x) = 0$.

Answer: (2 points) By the continuity of probability measures

$$P(\{\omega : X(\omega) = x\}) = \lim_{y \uparrow x} P(\{\omega : y < X(\omega) \leq x\}) = F(x) - \lim_{y \uparrow x} F(y) = F(x) - F(x-).$$

But $F(x) - F(x-) > 0$ if, and only if, F has a jump discontinuity at x.

5. Consider the following statement:*f is continuous almost everywhere if, and only if, it is almost everywhere equal to an everywhere continuous function.* Is this true or false? Explain, with precise mathematical arguments.

Answer: (3 points) False. Consider the function $I_{\mathbb{Q}}(x)$, where $x \in \mathbb{R}$. This function is nowhere continuous in $\mathbb{R}$, but it is equal to 0 almost everywhere, an everywhere continuous function. Alternatively, the function $I_{[0,\infty)}(x)$ is continuous everywhere except at $\{0\}$, a set of measure zero. So, it is continuous almost everywhere. However, there is no everywhere continuous function in $\mathbb{R}$ that is equal $I_{[0,\infty)}(x)$ almost everywhere.

6. Prove Theorem 3.21 in your notes.

Answer: (4 points) Suppose h is a simple function with $h(x) = \sum_{j=1}^{m} y_j I_{A_j}$ where $A_j = \{x \in \mathbb{R} : h(x) = y_j\}$. Then, since X has a density f_X

$$\int_{\mathbb{R}} h dP_X = \sum_{j=1}^{m} y_j P_X(A_j) = \sum_{j=1}^{m} y_j \int_{A_j} f_X(x) d\lambda(x) = \int_{\mathbb{R}} \sum_{j=1}^{m} y_j I_{A_j} f_X(x) d\lambda(x) = \int_{\mathbb{R}} h(x) f_X(x) d\lambda(x).$$

If h is a non-negative, by Theorem 3.3 in your notes there exists a sequence of non-negative simple functions $h_n \to h$ as $n \to \infty$ and $h_n \circ X \to h \circ X$. By Lebesgue's Monotone Convergence Theorem

$$\begin{aligned}\int_{\mathbb{R}} \lim_{n\to\infty} h_n dP_X &= \lim_{n\to\infty} \int_{\mathbb{R}} h_n dP_X = \lim_{n\to\infty} \int_{\mathbb{R}} h_n(x) f_X(x) d\lambda(x) = \int_{\mathbb{R}} \lim_{n\to\infty} h_n(x) f_X(x) d\lambda(x) \\ &= \int_{\mathbb{R}} h(x) f_X(x) d\lambda(x)\end{aligned}$$

If h is an integrable function, write $h = h^+ - h^-$ and repeat the previous case (h non-negative) for h^+ and h^-.

7. Adapt the proof of Lebesgue's Dominated Convergence Theorem in your notes to show that any sequence $\{f_n\}_{n\in\mathbb{N}}$ of measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ and $|f_n| \leq g$ for some g with g^p nonnegative and integrable satisfies

$$\lim_{n\to\infty} \int |f_n - f|^p d\mu = 0.$$

Answer: (3 points) First, note that $|f_n - f|^p \leq (|f_n| + |f|)^p$. Since $|f_n - f| \to 0$ we have that $|f_n| \to |f|$. Consequently, for all $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for $n \geq N_\epsilon$ we have

$$|f_n| - \epsilon \leq |f| \leq |f_n| + \epsilon \leq g + \epsilon$$

since $|f_n| < g$. Consequently, $|f| \leq g$, $|f|^p \leq g^p$ and $|f_n - f|^p \leq 2^p g^p$ where g^p is nonnegative and integrable. Now, letting $\phi_n = |f_n - f|^p$ we have that $\lim_{n\to\infty} \phi_n = 0$ and by Lebesgue's dominated convergence theorem in the class notes

$$\lim_{n\to\infty} \int_{\mathbb{X}} \phi_n d\mu = \int_{\mathbb{X}} \lim_{n\to\infty} \phi_n d\mu = 0.$$

8. Let λ be the one-dimensional Lebesgue measure for the Borel sets of $\mathbb{R}$. Show that for every integrable function f, the function

$$g(x) = \int_{(0,x)} f(t) d\lambda, \text{ for } x > 0$$

is continuous.

Answer: (3 points) Consider a sequence $\{y_n\}_{n\in\mathbb{N}}$ with $0 < x < y_n$ such that $\lim_{n\to\infty} y_n = x$. Then,

$$\begin{aligned} g(y_n) - g(x) &= \int_{(0,y_n)} f d\lambda - \int_{(0,x)} f d\lambda = \int_{(0,\infty)} I_{(0,y_n)} f d\lambda - \int_{(0,\infty)} I_{(0,x)} f d\lambda \\ &= \int_{(0,\infty)} (I_{(0,y_n)} - I_{(0,x)}) f d\lambda = \int_{(0,\infty)} I_{(x,y_n)} f d\lambda \\ |g(y_n) - g(x)| &\leq \int_{(0,\infty)} I_{[x,y_n)} |f| d\lambda. \end{aligned}$$

Now, $I_{[x,y_n)}|f| \leq |f|$ and $\int_{(0,\infty)} |f| d\lambda < \infty$ since f is integrable. Also, $\lim_{n\to\infty} I_{[x,y_n)} f = 0$ almost everywhere (ae). Thus, by dominated convergence in the class notes

$$\begin{aligned} \lim_{n\to\infty} |g(y_n) - g(x)| &\leq \lim_{n\to\infty} \int_{(0,\infty)} I_{(x,y_n)} |f| d\lambda \\ &= \int_{(0,\infty)} \lim_{n\to\infty} I_{(x,y_n)} |f| d\lambda = 0. \end{aligned}$$

By repeating the argument for $y_n \uparrow x$ we obtain continuity of g at x.

9. Show that if X is a random variable with $E(|X|^p) < \infty$ then $|X|$ is almost everywhere real valued.

Answer: (4 points) Let $N = \{\omega : |X(\omega)| = \infty\} = \{\omega : |X(\omega)|^p = \infty\}$. Then $N = \cap_{n\in\mathbb{N}} \{\omega : |X(\omega)|^p \geq n\}$. Then,

$$\begin{aligned} P(N) &= P\left(\cap_{n\in\mathbb{N}} \{\omega : |X(\omega)|^p \geq n\}\right) \\ &= \lim_{n\to\infty} P\left(\{\omega : |X(\omega)|^p \geq n\}\right) \text{ by continuity of probability measures} \\ &\leq \lim_{n\to\infty} \frac{1}{k} \int_{\Omega} |X|^p dP \text{ by Markov's Inequality} \\ &= 0 \text{ since } \textstyle\int_{\Omega} |X|^p dP \text{ is finite.} \end{aligned}$$

10. Suppose $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B})$ is a random variable with $E(|X|) < \infty$. Let $N \in \mathcal{F}$ be such that $P(N) = 0$ and define

$$Y(\omega) = \begin{cases} X(\omega) & \text{if } \omega \notin N \\ c & \text{if } \omega \in N \end{cases},$$

where $c \in \mathbb{R}$. Is Y integrable? Is $E(X) = E(Y)$?

Answer: (2 points) Yes, for both questions. We can change an integrable random variables at any set of measure zero without changing the integral. This results from Theorem 3.12 in the class notes.

11. Let f be a density for the random variable X and $a > 0$. Show that

$$\frac{1}{a} P(f(X) < a) \le C$$

for some constant $C > 0$.

Answer: Let $A_a = \{x : f(x) < a\}$ and $A = \{x : \|x\| \le B\}$, where $\|x\| = \left(\sum_{i=1}^k x_i^2\right)^{1/2}$. Now, $A_a = (A_a \cap A) \cup (A_a \cap A^c)$ and

$$P(A_a) = P(A_a \cap A) + P(A_a \cap A^c) \le P(A_a \cap A) + P(A^c).$$

Now, $P(A_a \cap A) = \int_{A_a \cap A} f(x)dx$ since f is a density. But over $A_a \cap A$, $f(x) < a$, so

$$P(A_a \cap A) \le a \int_{A_a \cap A} dx \le b \int_A dx.$$

Now, $\|x\| \le B$ implies $|x_i| \le B$. So, $\int_A dx \le \int_{|x_1| \le B} \cdots \int_{|x_k| \le B} dx = (2B)^k$ and we have $P(A_a \cap A) \le a(2B)^k$. So,

$$P(A_a) \le b(2B)^k + P(A^c).$$

Now, for any $\epsilon > 0$, $P(A^c) = \int_{\|x\| > B} f(x)dx < \epsilon$ for B sufficiently large, since $\int f(x)dx = 1$. Then,

$$P(A_a) \le a(2B)^k + \epsilon,$$

which implies $P(A_a) \le a(2B)^k$.