Econ 7818, Homework 2, part 2, Professor Martins.
Due date: Parts 1 and 2 of Homework 2 are due October 17 by 5:30 PM in my mailbox.

1. Prove Theorem 4.2 in your notes.

   Answer: Let $f = \sum_{i=0}^{I} y_i I_{A_i}$ and $g = \sum_{j=0}^{J} z_j I_{B_j}$ be standard representations of f and g. Then,

   $$f \pm g = \sum_{i=0}^{I}\sum_{j=0}^{J}(y_i \pm z_j) I_{A_i \cap B_j}$$

   and

   $$fg = \sum_{i=0}^{I}\sum_{j=0}^{J}(y_i z_j) I_{A_i \cap B_j},$$

   with $(A_i \cap B_j) \cap (A_{i'} \cap B_{j'}) = \emptyset$ whenever $(i,j) \neq (i',j')$. After relabeling and merging the double sums into single sums we have the result.

   The case for cf is obvious.

   f simple implies f^+ and f^- are simple by definition, and since $|f| = f^+ + f^-$, $|f|$ is simple.

2. Prove Theorem 4.9 in your notes.

   Answer: Since $f = f^+ - f^-$ and f^+ and f^- are nonnegative, use Theorems 4.6 and 4.8 in your notes.

3. Use Markov's inequality in your notes to prove the following for $a > 0$ and $g : (0,\infty) \to (0,\infty)$ that is increasing:

   $$P(|X(\omega)| \geq a) \leq \frac{1}{g(a)} \int g(|X|) dP$$

   Answer: Since g is increasing, $\{\omega : |X(\omega)| \geq a\} = \{\omega : g(|X(\omega)|) \geq g(a)\}$. Hence, since g is positive

   $$g(a) I_{\{\omega : |X(\omega)| \geq a\}} = g(a) I_{\{\omega : g(|X(\omega)|) \geq g(a)\}} \leq g(|X(\omega)|).$$

   Integrating both sides we have $g(a)P(\{\omega : |X(\omega)| \geq a\}) \leq \int g(|X(\omega)|) dP$. This completes the proof as $g(a) > 0$.

4. Let X be a random variable defined in the probability space $(\Omega, \mathcal{F}, P)$ with $E(X^2) < \infty$. Consider a function $f : \mathbb{R} \to \mathbb{R}$. What restrictions are needed on f to guarantee that $f(X)$ is a random variable with $E(f(X)^2) < \infty$?

   Answer: Recall that if $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, we say that X is a random variable (measurable real valued function) if, and only if, for all $B \in \mathcal{B}_{\mathbb{R}}$ we have $X^{-1}(B) \in \mathcal{F}$. Hence, if $h(\omega) := f(X(\omega)) = (f \circ X)(\omega) : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ we require that for all $B \in \mathcal{B}_{\mathbb{R}}$ we have $h^{-1}(B) = (f \circ X)^{-1}(B) = X^{-1}(f^{-1}(B)) \in \mathcal{F}$. That is, $f^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$.

   Since X is a random variable (measurable) and given that $f^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$ for all $B \in \mathcal{B}_{\mathbb{R}}$, $f(X)$ is a random variable (measurable). Since the f^2 is a continuous function of f, f^2 is also a random variable (measurable). Hence, we can consider the integrability (or not) of $f(X)^2$, i.e., whether or not $E(f(X)^2) < \infty$. We give two general restrictions on f that give $E(f(X)^2) < \infty$. First, suppose that $\sup_{\omega \in \Omega} |h(\omega)| = \sup_{\omega \in \Omega} |(f \circ X)(\omega)| < C$. Then,

   $$\left| \int f^2 dP \right| \leq \int h^2 dP \leq C^2 \int dP = C^2.$$

Second, suppose that $h^2 \leq X^2$ for all $\omega \in \Omega$. Then, $\int h^2 dP \leq \int X^2 dP < \infty$.

Note that, in general, it is not true that $E(f(X)^2) < \infty$ even if $E(X^2) < \infty$. For example, suppose that $X \sim U[0,1]$. Then, $E(X^2) = 1/3$. Now, let $Y := f(X) = \tan\left(\pi(X - \frac{1}{2})\right)$ and we can easily obtain that the probability density of Y is

$$f_Y(y) = \left|\frac{d}{dy} f^{-1}(y)\right| = \left|\frac{d}{dy}\left(\frac{1}{2} + \frac{1}{\pi}\arctan(y)\right)\right| = \frac{1}{\pi}\frac{1}{1+y^2},\ y \in \mathbb{R}.$$

But this is the Cauchy density and $\int y^2 f_Y(y) dy$ does not exist.

5. Let $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B})$ be a random variable. Show that if $V(X) := E\left((X - E(X))\right)^2 = 0$ then X is a constant with probability 1.

   Answer: (2 points) From your notes, if $\int_\Omega X^2 dP = 0$ then $X^2 = 0$ almost everywhere. If N is a null set $\int_\Omega X^2 dP = \int_N X^2 dP + \int_{N^c} X^2 dP = \int_N X^2 dP + \int_{N^c} 0 dP = 0$. Thus, $P(X^2 = x) = 0$ for $x \neq 0$ and $P(X^2 = 0) = 1$. But this is equivalent to $P(X = 0) = 1$. Hence, $V(X) = E\left((X - E(X))\right)^2 = 0$ implies $P(X - E(X) = 0) = P(X = E(X)) = 1$.

6. Adapt the proof of Lebesgue's Dominated Convergence Theorem in your notes to show that any sequence $\{f_n\}_{n\in\mathbb{N}}$ of measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ and $|f_n| \leq g$ for some g with g^p nonnegative and integrable satisfies

$$\lim_{n\to\infty} \int |f_n - f|^p d\mu = 0.$$

   Answer: First, note that $|f_n - f|^p \leq (|f_n| + |f|)^p$. Since $|f_n - f| \to 0$ we have that $|f_n| \to |f|$. Consequently, for all $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for $n \geq N_\epsilon$ we have

$$|f_n| - \epsilon \leq |f| \leq |f_n| + \epsilon \leq g + \epsilon$$

   since $|f_n| < g$. Consequently, $|f| \leq g$, $|f|^p \leq g^p$ and $|f_n - f|^p \leq 2^p g^p$ where g^p is nonnegative and integrable. Now, letting $\phi_n = |f_n - f|^p$ we have that $\lim\limits_{n\to\infty} \phi_n = 0$ and by Lebesgue's dominated convergence theorem in the class notes

$$\lim_{n\to\infty} \int_{\mathbb{X}} \phi_n d\mu = \int_{\mathbb{X}} \lim_{n\to\infty} \phi_n d\mu = 0.$$

7. Let λ be the one-dimensional Lebesgue measure for the Borel sets of $\mathbb{R}$. Show that for every integrable function f, the function

$$g(x) = \int_{(0,x)} f(t) d\lambda, \text{ for } x > 0$$

   is continuous.

   Answer: Consider a sequence $\{y_n\}_{n\in\mathbb{N}}$ with $0 < x < y_n$ such that $\lim\limits_{n\to\infty} y_n = x$. Then,

$$\begin{aligned} g(y_n) - g(x) &= \int_{(0,y_n)} f d\lambda - \int_{(0,x)} f d\lambda = \int_{(0,\infty)} I_{(0,y_n)} f d\lambda - \int_{(0,\infty)} I_{(0,x)} f d\lambda \\ &= \int_{(0,\infty)} (I_{(0,y_n)} - I_{(0,x)}) f d\lambda = \int_{(0,\infty)} I_{(x,y_n)} f d\lambda \\ |g(y_n) - g(x)| &\leq \int_{(0,\infty)} I_{[x,y_n)} |f| d\lambda. \end{aligned}$$

Now, $I_{[x,y_n)}|f| \leq |f|$ and $\int_{(0,\infty)} |f| d\lambda < \infty$ since f is integrable. Also, $\lim_{n\to\infty} I_{[x,y_n)} f = 0$ almost everywhere (ae). Thus, by dominated convergence in the class notes

$$\begin{aligned} \lim_{n\to\infty} |g(y_n) - g(x)| &\leq \lim_{n\to\infty} \int_{(0,\infty)} I_{(x,y_n)} |f| d\lambda \\ &= \int_{(0,\infty)} \lim_{n\to\infty} I_{(x,y_n)} |f| d\lambda = 0. \end{aligned}$$

By repeating the argument for $y_n \uparrow x$ we obtain continuity of g at x.

8. Show that if X is a random variable with $E(|X|^p) < \infty$ then $|X|$ is almost everywhere real valued.

   Answer: Let $N = \{\omega : |X(\omega)| = \infty\} = \{\omega : |X(\omega)|^p = \infty\}$. Then $N = \cap_{n\in\mathbb{N}}\{\omega : |X(\omega)|^p \geq n\}$. Then,

$$\begin{aligned} P(N) &= P\left(\cap_{n\in\mathbb{N}}\{\omega : |X(\omega)|^p \geq n\}\right) \\ &= \lim_{n\to\infty} P\left(\{\omega : |X(\omega)|^p \geq n\}\right) \text{ by continuity of probability measures} \\ &\leq \lim_{n\to\infty} \frac{1}{k} \int_\Omega |X|^p dP \text{ by Markov's Inequality} \\ &= 0 \text{ since } \textstyle\int_\Omega |X|^p dP \text{ is finite.} \end{aligned}$$

9. Suppose $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B})$ is a random variable with $E(|X|) < \infty$. Let $N \in \mathcal{F}$ be such that $P(N) = 0$ and define

$$Y(\omega) = \begin{cases} X(\omega) & \text{if } \omega \notin N \\ c & \text{if } \omega \in N \end{cases},$$

   where $c \in \mathbb{R}$. Is Y integrable? Is $E(X) = E(Y)$?

   Answer: Yes, for both questions. We can change an integrable random variables at any set of measure zero without changing the integral.

10. Prove Theorem 5.13 in your notes.