Econ 7818, Homework 3 - part 2, Professor Martins. Due date: November 17th in class. This due date also applies to part 1 of Homework 3.

1. Show that if $Y_n \overset{d}{\to} Y$ then $Y_n = O_p(1)$.

   Answer: Without loss of generality let $a > 0$. Provided that a and $-a$ are continuity points of F_Y, we can write that,$P(|Y_n| > a) \to P(|Y| > a)$ as $n \to \infty$. Hence, for every $\epsilon > 0$ there exists N_ϵ such that,

   $$|P(|Y_n| > a) - P(|Y| > a)| < \epsilon \text{ for all } n \geq N_\epsilon$$

   or

   $$P(|Y| > a) - \epsilon < P(|Y_n| > a) < P(|Y| > a) + \epsilon.$$

   We can choose a such that $P(|Y| > a) < \delta$ for any $\delta > 0$. Thus, $P(|Y_n| > a) < \delta + \epsilon$ for all $n \geq N_\epsilon$.

2. Let $g : S \subseteq \mathbb{R}$ be continuous on S, and X_t and X_s be random variables defined on $(\Omega, \mathcal{F}, P)$ taking values in S. Show that: a) if X_t is independent of X_s, then $g(X_t)$ is independent of $g(X_s)$; b) if X_t and X_s are identically distributed, then $g(X_t)$ and $g(X_s)$ are identically distributed.

   Answer: Let $Y_t = g(X_t)$ and $Y_s = g(X_s)$. g continuous assures that both Y_t and Y_s are random variables.

   a) $F_{Y_t,Y_s}(a,b) = P(S = \{\omega : Y_t \leq a \text{ and } Y_s \leq b\})$. Let $S_t = \{X_t(\omega) : Y_t(\omega) \leq a\}, S_s = \{X_s(\omega) : Y_s(\omega) \leq b\}$. Since, $S = S_t \cap S_s$ and by independence $P(S) = P(S_t)P(S_s)$ which implies $F_{Y_t,Y_s}(a,b) = F_{Y_t}(a)F_{Y_s}(b)$.

   b) $F_{Y_t}(a) = P(S_t) = P(\{X_s(\omega) : Y_s(\omega) \leq a\}) = F_{Y_s}(a)$.

3. Let $\{X_n\}$ be a sequence of independent random variables that converges in probability to a limit X. Show that X is almost surely a constant.

   Answer: Recall that if X is almost surely a constant, say c, $P(\{\omega : X(\omega) \neq c\}) = 0$. Then, the distribution function F associated with X is given by

   $$F(x) = \begin{cases} 0, & \text{if } x < c \\ 1, & \text{if } x \geq c \end{cases} .$$

   If X is not a constant, there exists a c and $0 < \epsilon < 1/2$ such that $P(X < c) > 2\epsilon$ and $P(X \leq c + \epsilon) < 1 - 2\epsilon$ or $P(X > c + \epsilon) > 2\epsilon$. Since $X_n \overset{p}{\to} X$ than $X_n \overset{d}{\to} X$. Consequently, for n sufficiently large and c a point of continuity of F we have

   $$F(c) - \epsilon < F_n(c) < F(c) + \epsilon$$

   which implies that $\epsilon < F_n(c)$. Also, $1 - F_n(c + \epsilon) > 1 - F(c + \epsilon) - \epsilon$ which implies $P(X_n > c + \epsilon) > P(X > c + \epsilon) - \epsilon > \epsilon$. Since $X_n \overset{p}{\to} X$, for n sufficiently large $P(\{\omega : |X_r - X_s| > \epsilon\}) < \epsilon^3$. Since $\{\omega : |X_r - X_s| > \epsilon\} = \{\omega : X_r - X_s > \epsilon\} \cup \{\omega : X_r - X_s < -\epsilon\}$ we note that if $X_r < c$ and $X_s > c + \epsilon$ then $X_r - X_s > \epsilon$ is equivalent to $X_r - X_s < -\epsilon$. Consequently,

   $$P(\{\omega : |X_r - X_s| > \epsilon\}) \leq P(\{\omega : X_r < c \text{ and } X_s > c + \epsilon\}).$$

But since X_r and X_s are independent $P(\{\omega : X_r < c \text{ and } X_s > c+\epsilon\}) = P(\{\omega : X_r < c\})P(\{\omega : X_s > c+\epsilon\}) > \epsilon^2$. Hence,

$$\epsilon^3 > P(\{\omega : |X_r - X_s| > \epsilon\}) > \epsilon^2,$$

a contradiction.

4. Suppose $\frac{X_n-\mu}{\sigma_n} \xrightarrow{d} Z$ where the non-random sequence $\sigma_n \to 0$ as $n \to \infty$, and g is a function which is differentiable at μ. Then, show that $\frac{g(X_n)-g(\mu)}{g^{(1)}(\mu)\sigma_n} \xrightarrow{d} Z$.

   Answer: From question 2, if $Z_n \xrightarrow{d} Z$ then $Z_n = O_p(1)$. Let $Z_n = \frac{X_n-\mu}{\sigma_n}$ and write $X_n = \mu + \sigma_n Z_n = \mu + O_p(\sigma_n)$. By Taylor's Theorem

$$\frac{1}{\sigma_n} g(X_n) - g(\mu) = g^{(1)}(\mu)\frac{(X_n-\mu)}{\sigma_n} + o_p(1).$$

   Since $\frac{X_n-\mu}{\sigma_n} \xrightarrow{d} Z$, we have the result.

5. Show that if $\{X_j\}_{j\in\mathbb{N}}$ be a sequence of random variables with $E(X_j) = 0$ and $\sum_{j=1}^{\infty} \frac{1}{a_j^p} E(|X_j|^p) < \infty$ for some $p \geq 1$ and a sequence of positive constants $\{a_j\}_{j\in\mathbb{N}}$. Then,

$$\sum_{j=1}^{\infty} P(|X_j| > a_j) < \infty \text{ and } \sum_{j=1}^{\infty} \frac{1}{a_j} |E(X_j I_{\{\omega:|X_j|\leq a_j\}})| < \infty.$$

   Furthermore, for any $r \geq p$,

$$\sum_{j=1}^{\infty} \frac{1}{a_j^r} E(|X_j|^r I_{\{\omega:|X_j|\leq a_j\}}) < \infty.$$

   Use this result to prove Theorem 6.3 in your class notes.

   Answer: Note that

$$P(\{\omega : |X_j| > a_j\}) = 1 - P(\{\omega : |X_j| \leq a_j\}) = \int_\Omega \left(1 - I_{\{\omega:|X_j|\leq a_j\}}\right) dP.$$

   If $\omega \in \{\omega : |X_j| \leq a_j\}$, then $P(\{\omega : |X_j| > a_j\}) = 0$. If $|X_j| > a_j$, then $|X_j|^p > a_j^p$ and $|X_j|^p/a_j^p > 1$. Hence,

$$P(\{\omega : |X_j| > a_j\}) < \int_\Omega |X_j|^p/a_j^p dP = \frac{1}{a_j^p} E\left(|X_j|^p\right)$$

   and

$$\sum_{j=1}^{\infty} P(\{\omega : |X_j| > a_j\}) < \sum_{j=1}^{\infty} \frac{1}{a_j^p} E\left(|X_j|^p\right) < \infty.$$

   Now,

$$\begin{aligned}
\frac{1}{a_j}|E(X_j I_{\{\omega:|X_j|\leq a_j\}})| &= \frac{1}{a_j}|E(X_j) - E(X_j I_{\{\omega:|X_j|\leq a_j\}})|, \text{ since } E(X_j) = 0. \\
&\leq \frac{1}{a_j} E\left(|X_t|(1 - I_{\{\omega:|X_j|\leq a_j\}}\right) \\
&\leq \frac{1}{a_j^p} E\left(|X_j|^p(1 - I_{\{\omega:|X_j|\leq a_j\}}\right) \text{ since } \tfrac{|X_j|^p}{a_j^p} \geq \tfrac{|X_j|}{a_j} \text{ if } p \geq 1. \\
&\leq \frac{1}{a_j^p} E\left(|X_j|^p\right).
\end{aligned}$$

Hence,

$$\sum_{j=1}^{\infty} \frac{1}{a_j} |E(X_j I_{\{\omega:|X_j| \leq a_j\}})| < \sum_{j=1}^{\infty} \frac{1}{a_j^p} E\left(|X_j|^p\right) < \infty.$$

Lastly, if $|X_j| \leq a_j$ we have that $\frac{1}{a_j}|X_j| \leq 1$. Then, for $r \geq p \geq 1$

$$\frac{1}{a_j^r}|X_j|^r I_{\{\omega:|X_j| \leq a_j\}} \leq \frac{1}{a_j^p}|X_j|^p I_{\{\omega:|X_j| \leq a_j\}} \leq \frac{1}{a_j}|X_j| I_{\{\omega:|X_j| \leq a_j\}}$$

and

$$E\left(\frac{1}{a_j^r}|X_j|^r I_{\{\omega:|X_j| \leq a_j\}}\right) \leq E\left(\frac{1}{a_j^p}|X_j|^p I_{\{\omega:|X_j| \leq a_j\}}\right).$$

Hence,

$$\sum_{j=1}^{\infty} E\left(\frac{1}{a_j^r}|X_j|^r I_{\{\omega:|X_j| \leq a_j\}}\right) < \infty.$$

In Theorem 6.3, the sequence of random variables $\{X_j\}_{j \in \mathbb{N}}$ is independent and has expectation μ_j. Hence, if $W_j := X_j - \mu_j$, we have $E(W_j) = 0$. Furthermore, in Theorem 6.3 it is assumed that for some $\delta > 0$

$$\sum_{j=1}^{\infty} \frac{E(|W_j|^{1+\delta})}{j^{1+\delta}} < \infty.$$

Now, note that for any $n \in \mathbb{N}$ we have $\sum_{j=1}^{n} \frac{E(|W_j|^{1+\delta})}{n^{1+\delta}} \leq \sum_{j=1}^{n} \frac{E(|W_j|^{1+\delta})}{j^{1+\delta}}$ and

$$\lim_{n \to \infty} \sum_{j=1}^{n} \frac{E(|W_j|^{1+\delta})}{n^{1+\delta}} \leq \lim_{n \to \infty} \sum_{j=1}^{n} \frac{E(|W_j|^{1+\delta})}{j^{1+\delta}} < \infty.$$

Now, in the first part of this answer, take $a_j = n$ for all j and for any $r > 1 + \delta$. Then, we have

$$\sum_{j=1}^{\infty} P(|W_j| > n) < \infty \text{ and } \sum_{j=1}^{\infty} \frac{1}{n^r} E(|W_j|^r I_{\{\omega:|W_j| \leq n\}}) < \infty.$$

Hence, taking $r = 2$ the conditions on Theorem 6.2 are met and we have

$$\frac{1}{n}\sum_{j=1}^{n} W_j - \frac{1}{n}\sum_{i=1}^{n} E\left(W_j I_{\{\omega:|W_j| \leq n\}}\right) = \frac{1}{n}\sum_{j=1}^{n}(X_j - \mu_j) - \frac{1}{n}\sum_{i=1}^{n} E\left(W_j I_{\{\omega:|W_j| \leq n\}}\right) = o_p(1).$$

But since $E(W_j) = 0$, we have $E\left(W_j I_{\{\omega:|W_j| \leq n\}}\right) \to 0$ as $n \to \infty$. Thus, $\frac{1}{n}\sum_{j=1}^{n}(X_j - \mu_j) = o_p(1)$.