Econ 7818, Homework 3 - part 2, Professor Martins. Due date: November 30th in class. This due date also applies to part 1 of Homework 3.

1. Show that if $Y_n \xrightarrow{d} Y$ then $Y_n = O_p(1)$.

   Answer: Without loss of generality let $a > 0$. Provided that a and $-a$ are continuity points of F_Y, we can write that,$P(|Y_n| > a) \to P(|Y| > a)$ as $n \to \infty$. Hence, for every $\epsilon > 0$ there exists N_ϵ such that,

   $$|P(|Y_n| > a) - P(|Y| > a)| < \epsilon \text{ for all } n \geq N_\epsilon$$

   or

   $$P(|Y| > a) - \epsilon < P(|Y_n| > a) < P(|Y| > a) + \epsilon.$$

   We can choose a such that $P(|Y| > a) < \delta$ for any $\delta > 0$. Thus, $P(|Y_n| > a) < \delta + \epsilon$ for all $n \geq N_\epsilon$.

2. Let $g : S \subseteq \mathbb{R}$ be continuous on S, and X_t and X_s be random variables defined on $(\Omega, \mathcal{F}, P)$ taking values in S. Show that: a) if X_t is independent of X_s, then $g(X_t)$ is independent of $g(X_s)$; b) if X_t and X_s are identically distributed, then $g(X_t)$ and $g(X_s)$ are identically distributed.

   Answer: Let $Y_t = g(X_t)$ and $Y_s = g(X_s)$. g continuous assures that both Y_t and Y_s are random variables.

   a) $F_{Y_t,Y_s}(a,b) = P(S = \{\omega : Y_t \leq a \text{ and } Y_s \leq b\})$. Let $S_t = \{X_t(\omega) : Y_t(\omega) \leq a\}, S_s = \{X_s(\omega) : Y_s(\omega) \leq b\}$. Since, $S = S_t \cap S_s$ and by independence $P(S) = P(S_t)P(S_s)$ which implies $F_{Y_t,Y_s}(a,b) = F_{Y_t}(a)F_{Y_s}(b)$.

   b) $F_{Y_t}(a) = P(S_t) = P(\{X_s(\omega) : Y_s(\omega) \leq a\}) = F_{Y_s}(a)$.

3. Let $\{X_n\}$ be a sequence of independent random variables that converges in probability to a limit X. Show that X is almost surely a constant.

   Answer: Recall that if X is almost surely a constant, say c, $P(\{\omega : X(\omega) \neq c\}) = 0$. Then, the distribution function F associated with X is given by

   $$F(x) = \begin{cases} 0, & \text{if } x < c \\ 1, & \text{if } x \geq c \end{cases} .$$

   If X is not a constant, there exists a c and $0 < \epsilon < 1/2$ such that $P(X < c) > 2\epsilon$ and $P(X \leq c + \epsilon) < 1 - 2\epsilon$ or $P(X > c + \epsilon) > 2\epsilon$. Since $X_n \xrightarrow{p} X$ than $X_n \xrightarrow{d} X$. Consequently, for n sufficiently large and c a point of continuity of F we have

   $$F(c) - \epsilon < F_n(c) < F(c) + \epsilon$$

   which implies that $\epsilon < F_n(c)$. Also, $1 - F_n(c + \epsilon) > 1 - F(c + \epsilon) - \epsilon$ which implies $P(X_n > c + \epsilon) > P(X > c + \epsilon) - \epsilon > \epsilon$. Since $X_n \xrightarrow{p} X$, for n sufficiently large $P(\{\omega : |X_r - X_s| > \epsilon\}) < \epsilon^3$. Since $\{\omega : |X_r - X_s| > \epsilon\} = \{\omega : X_r - X_s > \epsilon\} \cup \{\omega : X_r - X_s < -\epsilon\}$ we note that if $X_r < c$ and $X_s > c + \epsilon$ then $X_r - X_s > \epsilon$ is equivalent to $X_r - X_s < -\epsilon$. Consequently,

   $$P(\{\omega : |X_r - X_s| > \epsilon\}) \leq P(\{\omega : X_r < c \text{ and } X_s > c + \epsilon\}).$$

But since X_r and X_s are independent $P(\{\omega : X_r < c \text{ and } X_s > c+\epsilon\}) = P(\{\omega : X_r < c\})P(\{\omega : X_s > c+\epsilon\}) > \epsilon^2$. Hence,

$$\epsilon^3 > P(\{\omega : |X_r - X_s| > \epsilon\}) > \epsilon^2,$$

a contradiction.

4. Suppose $\frac{X_n-\mu}{\sigma_n} \xrightarrow{d} Z$ where the non-random sequence $\sigma_n \to 0$ as $n \to \infty$, and g is a function which is differentiable at μ. Then, show that $\frac{g(X_n)-g(\mu)}{g^{(1)}(\mu)\sigma_n} \xrightarrow{d} Z$.

   Answer: From question 2, if $Z_n \xrightarrow{d} Z$ then $Z_n = O_p(1)$. Let $Z_n = \frac{X_n-\mu}{\sigma_n}$ and write $X_n = \mu + \sigma_n Z_n = \mu + O_p(\sigma_n)$. By Taylor's Theorem

$$\frac{1}{\sigma_n} g(X_n) - g(\mu) = g^{(1)}(\mu)\frac{(X_n-\mu)}{\sigma_n} + o_p(1).$$

   Since $\frac{X_n-\mu}{\sigma_n} \xrightarrow{d} Z$, we have the result.

5. Prove item 1 in Remark 7.1 on your class notes.

   Answer: For $\epsilon > 0$ we have that

$$\{\omega : |X_n + Y_n - X - Y| > \epsilon\} \subseteq \{\omega : |X_n - X| > \epsilon/2\} \cup \{\omega : |Y_n - Y| > \epsilon/2\}$$

   The probability of the events on the union on right-hand side go to zero as $n \to \infty$. By monotonicity of probability measures we have the results.
   For $\epsilon > 0$,

$$\begin{aligned} P(\{\omega : |X_nY_n - XY| > \epsilon\}) &= P\left(|(X_n - X)(Y_n - Y) + (X_n - X)Y + X(Y_n - Y)| > \epsilon\right) \\ &\le P\left(|(X_n - X)||(Y_n - Y)| > \epsilon/3\right) + P\left(|(X_n - X)||Y| > \epsilon/3\right) \\ &+ P\left(|X||(Y_n - Y)| > \epsilon/3\right) \end{aligned}$$

   Now, for any $\delta > 0$ we have that

$$P\left(|(X_n - X)||Y| > \epsilon/3\right) \le P\left(|(X_n - X)| > \frac{\epsilon}{3\delta}\right) + P\left(|Y| > \delta\right)$$

   which tends to zero as $n \to \infty$ and $\delta \to \infty$. Using the same argument for the other terms we have the result.

6. Show that if $\{X_n\}_{n\in\mathbb{N}}$ and X are random variables defined on the same probability space and $r > s \ge 1$ and $X_n \xrightarrow{\mathcal{L}_r} X$, then $X_n \xrightarrow{\mathcal{L}_s} X$.

   Answer: For arbitrary W let $Z = |W|^s$, $Y = 1$ and $p = r/s$. Then, by Hölder's Inequality

$$E|ZY| \le \|Z\|_p \|Y\|_{p/(p-1)}.$$

   Substituting Z and Y gives $E(|W|^s) \le E(|W|^{sp})^{1/p} = E(|W|^{s\frac{r}{s}})^{s/r}$. Raising both sides to $1/s$ gives

$$E(|W|^s)^{1/s} \le E(|W|^r)^{1/r}.$$

   Setting $W = X_n - X$ and taking limits as $n \to \infty$ gives the result.

7. Let U and V be two points in an n-dimensional unit cube, i.e., $[0,1]^n$ and X_n be the Euclidean distance between these two points which are chosen independently and uniformly. Show that $\frac{X_n}{\sqrt{n}} \xrightarrow{p} \frac{1}{\sqrt{6}}$.

   Answer: Let $U' = \begin{pmatrix} U_1 & \cdots & U_n \end{pmatrix}$ and $V' = \begin{pmatrix} V_1 & \cdots & V_n \end{pmatrix}$. Then, $X_n = \left(\sum_{i=1}^n (U_i - V_i)^2\right)^{1/2}$ and we can write

$$\frac{1}{n}E(X_n^2) = \frac{1}{n}\sum_{i=1}^n E((U_i - V_i)^2) = \int_0^1 \int_0^1 (u-v)^2 dudv = 1/6$$

   where the last equality follows from routine integration. Then, since $E(|(U-V)^2|) = E((U-V)^2) < \infty$, by Kolmogorov's Law of Large Numbers

$$\frac{1}{n}X_n^2 = \frac{1}{n}\sum_{i=1}^n (U_i - V_i)^2 \xrightarrow{p} 1/6.$$

   Since, $f(x) = x^{1/2}$ is a continuous function $[0,\infty)$, by Slutsky Theorem if $\frac{1}{n}X_n^2 \xrightarrow{p} 1/6$ then $f\left(\frac{1}{n}X_n^2\right) \xrightarrow{p} f(1/6)$. Consequently,

$$\frac{1}{\sqrt{n}}X_n \xrightarrow{p} 1/\sqrt{6}.$$