Econ 7818, Homework 3 - part 1, Professor Martins. Due date: TBA.

1. Let $\{g_n\}_{n=1,2,\ldots}$ be a sequence of real valued functions that converge uniformly to g on an open set S, containing x, and g is continuous at x. Show that if $\{X_n\}_{n=1,2,\ldots}$ is a sequence of random variables taking values in S such that $X_n \xrightarrow{p} X$, then

$$g_n(X_n) \xrightarrow{p} g(X).$$

Note: Recall that a sequence of real valued functions $\{g_n\}_{n=1,2,\ldots}$ converges uniformly to g on a set S if, for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ (depending only on ϵ) such that for all $n > N_\epsilon$, $|g_n(x) - g(x)| < \epsilon$ for every $x \in S$.

Answer: Let $\epsilon, \delta > 0$ and define the following subsets of the sample space: $S_1^n = \{\omega : |g_n(X_n) - g(X)| < \epsilon\}$, $S_2^n = \{\omega : |g_n(X_n) - g(X_n)| < \epsilon/2\}$, $S_3^n = \{\omega : |g(X_n) - g(X)| < \epsilon/2\}$, $S_4^n = \{\omega : X_n \in S\}$. By the triangle inequality, $S_1^n \supseteq S_2^n \cap S_3^n$. By continuity of g at X and openness of S, there exists γ_ϵ such that whenever $|X_n - X| < \gamma_\epsilon$, $|g(X_n) - g(X)| < \epsilon/2$ and $X_n \in S$. Letting, $S_5^n = \{\omega : |X_n - X| < \gamma_\epsilon\}$, we see that $S_5^n \subseteq S_3^n \cap S_4^n$. Since $X_n \xrightarrow{p} X$ and uniform convergence of g_n, there exists $N_{\delta,\epsilon}$ such that whenever $n > N_{\delta,\epsilon}$, $|g_n(X) - g(X)| < \epsilon/2$ for all $X \in S$ and $P(S_5^n) > 1 - \delta$. Thus, $n > N_{\delta,\epsilon}$ implies $S_4^n \subseteq S_2^n$. Consequently, $n > N_{\delta,\epsilon}$ implies $S_1^n \supseteq S_2^n \cap S_3^n \supseteq S_4^n \cap S_3^n \supseteq S_5^n$. Thus, $P(S_1^n) \geq P(S_5^n) > 1 - \delta$.

2. Show that $X_n \xrightarrow{as} X$ is equivalent to $P\left(\{\omega : \sup_{j \geq n} |X_j - X| \geq \epsilon\}\right) \to 0$ for all $\epsilon > 0$ as $n \to \infty$.

Answer: For any $\epsilon > 0$ and $k \in \mathbb{N}$ let $A_k(\epsilon) = \{\omega : |X_k(\omega) - X(\omega)| > \epsilon\}$. If for all $n \in \mathbb{N}$ we have that $P\left(\cup_{k>n} A_k(\epsilon)\right) > 0$ then it must be that $X_n \overset{as}{\nrightarrow} X$. Consequently,

$$\begin{aligned} X_n \xrightarrow{as} X \quad &\Leftrightarrow \quad \lim_{n\to\infty} P\left(\cup_{n<k} A_k(\epsilon)\right) = 0 \\ &\Leftrightarrow \quad P\left(\{\omega : \sup_{j\geq n} |X_j - X| > \epsilon\}\right) \to 0 \text{ as } n \to \infty. \end{aligned}$$

3. Prove item 1 of Remark 5.1 on your class notes.

Answer: For $\epsilon > 0$ we have that

$$\{\omega : |X_n + Y_n - X - Y| > \epsilon\} \subseteq \{\omega : |X_n - X| > \epsilon/2\} \cup \{\omega : |Y_n - Y| > \epsilon/2\}$$

The probability of the events on the union on right-hand side go to zero as $n \to \infty$. By monotonicity of probability measures we have the results.

For $\epsilon > 0$,

$$\begin{aligned} P(\{\omega : |X_nY_n - XY| > \epsilon\}) &= P\left(|(X_n - X)(Y_n - Y) + (X_n - X)Y + X(Y_n - Y)| > \epsilon\right) \\ &\leq P\left(|(X_n - X)||(Y_n - Y)| > \epsilon/3\right) + P\left(|(X_n - X)||Y| > \epsilon/3\right) \\ &+ P\left(|X||(Y_n - Y)| > \epsilon/3\right) \end{aligned}$$

Now, for any $\delta > 0$ we have that

$$P\left(|(X_n - X)||Y| > \epsilon/3\right) \leq P\left(|(X_n - X)| > \frac{\epsilon}{3\delta}\right) + P\left(|Y| > \delta\right)$$

which tends to zero as $n \to \infty$ and $\delta \to \infty$. Using the same argument for the other terms we have the result.

4. Let $n \in \mathbb{N}$ and $h_n > 0$ such that $h_n \to 0$ as $n \to \infty$. Show that if $\sum_{n=1}^{\infty} P(\{\omega : |X_n - X| \geq h_n\}) < \infty$ then $X_n \xrightarrow{p} X$.

   Answer: From question 2,

$$X_n \xrightarrow{as} X \quad \Leftrightarrow \quad \lim_{n\to\infty} P\left(\cup_{n<k} A_k(h_n)\right) = 0.$$

   But $P\left(\cup_{n<k} A_k(h_n)\right) \leq \sum_{k\geq n} P(A_k(\epsilon))$ and if $\sum_{n=1}^{\infty} P(\{\omega : |X_n - X| \geq h_n\}) < \infty$ then it must be that $\lim\limits_{n\to\infty} \sum_{k\geq n} P(A_k(\epsilon)) = 0$. Since convergence almost surely implies convergence in probability, the proof is complete.

5. Show that if $X_n \xrightarrow{p} X$ and $X_n \xrightarrow{p} Y$ then $P(\{\omega : X \neq Y\}) = 0$.

   Answer: Set the underlying probability space to be $(\Omega, \mathcal{F}, P)$. Note that $|X-Y| = |X-X_j+X_j-Y| \leq |X_j - X| + |X_j - Y|$. Consequently, for any $n \in \mathbb{N}$

$$\{\omega : |X - Y| > 2/n\} \subset \{\omega : |X_j - X| > 1/n\} \cup \{\omega : |X_j - Y| > 1/n\}$$

   Thus,

$$P(\{\omega : |X - Y| > 2/n\}) \leq P(\{\omega : |X_j - X| > 1/n\}) + P(\{\omega : |X_j - Y| > 1/n\})$$

   where the probabilities on the right-hand side of the inequality go to 0 as $j \to \infty$. That is, for all n, $\{\omega : |X - Y| > 2/n\}$ is a null set. But note that

$$\{\omega : X \neq Y\} \subset \cup_{n\in\mathbb{N}}\{\omega : |X - Y| > 2/n\} = \cup_{n\in\mathbb{N}}\{\omega : |X - Y| > 2/n\},$$

   which is a null set, completing the proof.

6. Suppose $\{X_i\}_{i=1}^{n}$ is a sequence of independent and identically distributed random variables. The distribution of these random variables has a density given by

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{if } x \notin [a, b] \end{cases}$$

   where $a, b \in \mathbb{R}$ with $b > a$. Consider the following random variables $a_n = \min\limits_{1\leq i\leq n} \{X_i\},\ b_n = \max\limits_{1\leq i\leq n} \{X_i\}$. Can you show that $a_n \xrightarrow{p} a$ and $b_n \xrightarrow{p} b$?

   Answer: $P(|\tilde{b}_n - b| < \epsilon) = P(-(\tilde{b}_n - b) < \epsilon) = P(\tilde{b}_n > b - \epsilon) = 1 - P(\tilde{b}_n \leq b - \epsilon)$. But

$$\begin{aligned} P(\max_{1\leq i\leq n} \{X_i\} \leq x) &= P(X_1 \leq x, \ldots, X_n \leq x) \\ &= P(X_1 \leq x)P(X_2 \leq x)\cdots P(X_n \leq x), \text{ by independence} \\ &= F(x)^n \text{ by the fact that the distribution is identical for all } X_i \end{aligned}$$

   So, $P(|\tilde{b}_n - b| < \epsilon) = 1 - (F(b - \epsilon))^n$. But since $0 < F(b - \epsilon) < 1$, as $n \to \infty$, $P(|\tilde{b}_n - b| < \epsilon) \to 1$.

7. Show that if $\{X_n\}_{n\in\mathbb{N}}$ and X are random variables defined on the same probability space and $r > s \geq 1$ and $X_n \xrightarrow{\mathcal{L}_r} X$, then $X_n \xrightarrow{\mathcal{L}_s} X$.

   Answer: For arbitrary W let $Z = |W|^s$, $Y = 1$ and $p = r/s$. Then, by Hölder's Inequality

$$E|ZY| \leq \|Z\|_p \|Y\|_{p/(p-1)}.$$

Substituting Z and Y gives $E(|W|^s) \leq E(|W|^{sp})^{1/p} = E(|W|^{s\frac{r}{s}})^{s/r}$. Raising both sides to $1/s$ gives

$$E(|W|^s)^{1/s} \leq E(|W|^r)^{1/r}.$$

Setting $W = X_n - X$ and taking limits as $n \to \infty$ gives the result.

8. Let U and V be two points in an n-dimensional unit cube, i.e., $[0,1]^n$ and X_n be the Euclidean distance between these two points which are chosen independently and uniformly. Show that $\frac{X_n}{\sqrt{n}} \xrightarrow{p} \frac{1}{\sqrt{6}}$.

   Answer: Let $U' = \begin{pmatrix} U_1 & \cdots & U_n \end{pmatrix}$ and $V' = \begin{pmatrix} V_1 & \cdots & V_n \end{pmatrix}$. Then, $X_n = \left(\sum_{i=1}^n (U_i - V_i)^2\right)^{1/2}$ and we can write

$$\frac{1}{n}E(X_n^2) = \frac{1}{n}\sum_{i=1}^n E((U_i - V_i)^2) = \int_0^1\int_0^1 (u-v)^2 dudv = 1/6$$

   where the last equality follows from routine integration. Then, since $E(|(U-V)^2|) = E((U-V)^2) < \infty$, by Kolmogorov's Law of Large Numbers

$$\frac{1}{n}X_n^2 = \frac{1}{n}\sum_{i=1}^n (U_i - V_i)^2 \xrightarrow{p} 1/6.$$

   Since, $f(x) = x^{1/2}$ is a continuous function $[0,\infty)$, by Slutsky Theorem if $\frac{1}{n}X_n^2 \xrightarrow{p} 1/6$ then $f\left(\frac{1}{n}X_n^2\right) \xrightarrow{p} f(1/6)$. Consequently,

$$\frac{1}{\sqrt{n}}X_n \xrightarrow{p} 1/\sqrt{6}.$$

9. Show that if a series converges absolutely, then it converges.

   Answer: Let $\{x_n\}_{n\in\mathbb{N}}$ and consider the series $\sum_{n=1}^\infty x_n$. The series converge absolutely if $\sum_{n=1}^\infty |x_n| < \infty$. Now, let $j < i$ and note that

$$b_i - b_j = \sum_{n=1}^i |x_n| - \sum_{n=1}^j |x_n| = \sum_{\ell=j+1}^i |x_\ell|.$$

   If $j \to \infty$, $b_i - b_j \to 0$ since $\sum_{n=1}^\infty |x_n| < \infty$. Now, since

$$\left|\sum_{\ell=j+1}^i x_\ell\right| \leq \sum_{\ell=j+1}^i |x_\ell| \to 0,$$

   and since $\mathbb{R}$ is complete $\sum_{\ell=1}^\infty x_\ell < \infty$.