Econ 7818, Homework 3 - part 1, Professor Martins. Due date: TBA.

1. Prove Theorem 5.13 in your notes.

   Answer: (4 points) Suppose h is a simple function with $h(x) = \sum_{j=1}^{m} y_j I_{A_j}$ where $A_j = \{x \in \mathbb{R} : h(x) = y_j\}$. Then, since X has a density f_X

$$\int_{\mathbb{R}} h dP_X = \sum_{j=1}^{m} y_j P_X(A_j) = \sum_{j=1}^{m} y_j \int_{A_j} f_X(x) d\lambda(x) = \int_{\mathbb{R}} \sum_{j=1}^{m} y_j I_{A_j} f_X(x) d\lambda(x) = \int_{\mathbb{R}} h(x) f_X(x) d\lambda(x).$$

   If h is a non-negative, by Theorem 3.3 in your notes there exists a sequence of non-negative simple functions $h_n \to h$ as $n \to \infty$ and $h_n \circ X \to h \circ X$. By Lebesgue's Monotone Convergence Theorem

$$\begin{aligned}\int_{\mathbb{R}} \lim_{n\to\infty} h_n dP_X &= \lim_{n\to\infty} \int_{\mathbb{R}} h_n dP_X = \lim_{n\to\infty} \int_{\mathbb{R}} h_n(x) f_X(x) d\lambda(x) = \int_{\mathbb{R}} \lim_{n\to\infty} h_n(x) f_X(x) d\lambda(x) \\ &= \int_{\mathbb{R}} h(x) f_X(x) d\lambda(x)\end{aligned}$$

   If h is an integrable function, write $h = h^+ - h^-$ and repeat the previous case (h non-negative) for h^+ and h^-.

2. Let f be a density for the random variable X and $a > 0$. Show that

$$\frac{1}{a} P(f(X) < a) \leq C$$

   for some constant $C > 0$.

   Answer: Let $A_a = \{x : f(x) < a\}$ and $A = \{x : |x| \leq B\}$. Now, $A_a = (A_a \cap A) \cup (A_a \cap A^c)$ and

$$P(A_a) = P(A_a \cap A) + P(A_a \cap A^c) \leq P(A_a \cap A) + P(A^c).$$

   Now, $P(A_a \cap A) = \int_{A_a \cap A} f(x) dx$ since f is a density. But over $A_a \cap A$, $f(x) < a$, so

$$P(A_a \cap A) \leq a \int_{A_a \cap A} d\lambda \leq a \int_A dx = a \int_{[-B,B]} d\lambda = a2B.$$

   So,

$$P(A_a) \leq a2B + P(A^c).$$

   Now, for any $\epsilon > 0$, $P(A^c) = \int_{|x|>B} f(x) d\lambda < \epsilon$ for B sufficiently large, since $\int f(x) d\lambda = 1$. Then,

$$P(A_a) \leq a2B + \epsilon,$$

   which implies $\frac{1}{a} P(A_a) \leq 2B := C$.

3. Give expressions for the distribution functions of $X^+(\omega) = \max\{X(\omega), 0\}$, $X^-(\omega) = -\min\{X(\omega), 0\}$ and $|X|$ in terms of the the distribution F of X.

   Answer: (3 points)

$$F_{X^+}(x) = P(\max\{X(\omega), 0\} \leq x) = \begin{cases} 0 & \text{if } x < 0, \\ F(x), & \text{if } x \geq 0. \end{cases}$$

$$F_{X^-}(x) = P\left(-\min\{X(\omega), 0\}, 0\} \le x\right) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - \lim_{y \uparrow -x} F(y), & \text{if } x \ge 0. \end{cases}$$

$$F_{|X|}(x) = P\left(|X| \le x\right) = \begin{cases} 0 & \text{if } x < 0, \\ F(x) - \lim_{y \uparrow -x} F(y), & \text{if } x \ge 0. \end{cases}$$

4. Show that the distribution F_X associated with the random variable X is continuous at x if, and only if, $P(X = x) = 0$.

   Answer: (2 points) By the continuity of probability measures

$$P(\{\omega : X(\omega) = x\}) = \lim_{y \uparrow x} P(\{\omega : y < X(\omega) \le x\}) = F(x) - \lim_{y \uparrow x} F(y) = F(x) - F(x-).$$

   But $F(x) - F(x-) > 0$ if, and only if, F has a jump discontinuity at x.

5. Adapt the proof of Lebesgue's Dominated Convergence Theorem in your notes to show that any sequence $\{f_n\}_{n \in \mathbb{N}}$ of measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ and $|f_n| \le g$ for some g with g^p nonnegative and integrable satisfies

$$\lim_{n\to\infty} \int |f_n - f|^p d\mu = 0.$$

   Answer: First, note that $|f_n - f|^p \le (|f_n| + |f|)^p$. Since $|f_n - f| \to 0$ we have that $|f_n| \to |f|$. Consequently, for all $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for $n \ge N_\epsilon$ we have

$$|f_n| - \epsilon \le |f| \le |f_n| + \epsilon \le g + \epsilon$$

   since $|f_n| < g$. Consequently, $|f| \le g$, $|f|^p \le g^p$ and $|f_n - f|^p \le 2^p g^p$ where g^p is nonnegative and integrable. Now, letting $\phi_n = |f_n - f|^p$ we have that $\lim\limits_{n\to\infty} \phi_n = 0$ and by Lebesgue's dominated convergence theorem in the class notes

$$\lim_{n\to\infty} \int_{\mathbb{X}} \phi_n d\mu = \int_{\mathbb{X}} \lim_{n\to\infty} \phi_n d\mu = 0.$$

6. Let $\{g_n\}_{n=1,2,\dots}$ be a sequence of real valued functions that converge uniformly to g on an open set S, containing x, and g is continuous at x. Show that if $\{X_n\}_{n=1,2,\dots}$ is a sequence of random variables taking values in S such that $X_n \xrightarrow{p} X$, then

$$g_n(X_n) \xrightarrow{p} g(X).$$

   Note: Recall that a sequence of real valued functions $\{g_n\}_{n=1,2,\dots}$ converges uniformly to g on a set S if, for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ (depending only on ϵ) such that for all $n > N_\epsilon$, $|g_n(x) - g(x)| < \epsilon$ for every $x \in S$.

   Answer: Let $\epsilon, \delta > 0$ and define the following subsets of the sample space: $S_1^n = \{\omega : |g_n(X_n) - g(X)| < \epsilon\}$, $S_2^n = \{\omega : |g_n(X_n) - g(X_n)| < \epsilon/2\}$, $S_3^n = \{\omega : |g(X_n) - g(X)| < \epsilon/2\}$, $S_4^n = \{\omega : X_n \in S\}$. By the triangle inequality, $S_1^n \supseteq S_2^n \cap S_3^n$. By continuity of g at X and openness of S, there exists γ_ϵ such that whenever $|X_n - X| < \gamma_\epsilon$, $|g(X_n) - g(X)| < \epsilon/2$ and $X_n \in S$. Letting, $S_5^n = \{\omega : |X_n - X| < \gamma_\epsilon\}$, we see that $S_5^n \subseteq S_3^n \cap S_4^n$. Since $X_n \xrightarrow{p} X$ and uniform convergence of g_n, there exists $N_{\delta,\epsilon}$ such that whenever $n > N_{\delta,\epsilon}$, $|g_n(X) - g(X)| < \epsilon/2$ for all $X \in S$ and $P(S_5^n) > 1 - \delta$. Thus, $n > N_{\delta,\epsilon}$ implies $S_4^n \subseteq S_2^n$. Consequently, $n > N_{\delta,\epsilon}$ implies $S_1^n \supseteq S_2^n \cap S_3^n \supseteq S_4^n \cap S_3^n \supseteq S_5^n$. Thus, $P(S_1^n) \ge P(S_5^n) > 1 - \delta$.

7. Show that $X_n \stackrel{as}{\to} X$ is equivalent to $P\left(\{\omega : \sup_{j\geq n} |X_j - X| \geq \epsilon\}\right) \to 0$ for all $\epsilon > 0$ as $n \to \infty$.

Answer: For any $\epsilon > 0$ and $k \in \mathbb{N}$ let $A_k(\epsilon) = \{\omega : |X_k(\omega) - X(\omega)| > \epsilon\}$. If for all $n \in \mathbb{N}$ we have that $P\left(\cup_{k>n} A_k(\epsilon)\right) > 0$ then it must be that $X_n \stackrel{as}{\nrightarrow} X$. Consequently,

$$\begin{aligned} X_n \stackrel{as}{\to} X \quad &\Leftrightarrow \quad \lim_{n\to\infty} P\left(\cup_{n<k} A_k(\epsilon)\right) = 0 \\ &\Leftrightarrow \quad P\left(\{\omega : \sup_{j\geq n} |X_j - X| > \epsilon\}\right) \to 0 \text{ as } n \to \infty. \end{aligned}$$

8. Let $n \in \mathbb{N}$ and $h_n > 0$ such that $h_n \to 0$ as $n \to \infty$. Show that if $\sum_{n=1}^{\infty} P(\{\omega : |X_n - X| \geq h_n\}) < \infty$ then $X_n \stackrel{p}{\to} X$.

Answer: From question 7,

$$X_n \stackrel{as}{\to} X \quad \Leftrightarrow \quad \lim_{n\to\infty} P\left(\cup_{n<k} A_k(h_n)\right) = 0.$$

But $P\left(\cup_{n<k} A_k(h_n)\right) \leq \sum_{k\geq n} P(A_k(\epsilon))$ and if $\sum_{n=1}^{\infty} P(\{\omega : |X_n - X| \geq h_n\}) < \infty$ then it must be that $\lim_{n\to\infty} \sum_{k\geq n} P(A_k(\epsilon)) = 0$. Since convergence almost surely implies convergence in probability, the proof is complete.

9. Show that if $X_n \stackrel{p}{\to} X$ and $X_n \stackrel{p}{\to} Y$ then $P(\{\omega : X \neq Y\}) = 0$.

Answer: Set the underlying probability space to be $(\Omega, \mathcal{F}, P)$. Note that $|X - Y| = |X - X_j + X_j - Y| \leq |X_j - X| + |X_j - Y|$. Consequently, for any $n \in \mathbb{N}$

$$\{\omega : |X - Y| > 2/n\} \subset \{\omega : |X_j - X| > 1/n\} \cup \{\omega : |X_j - Y| > 1/n\}$$

Thus,

$$P(\{\omega : |X - Y| > 2/n\}) \leq P(\{\omega : |X_j - X| > 1/n\}) + P(\{\omega : |X_j - Y| > 1/n\})$$

where the probabilities on the right-hand side of the inequality go to 0 as $j \to \infty$. That is, for all n, $\{\omega : |X - Y| > 2/n\}$ is a null set. But note that

$$\{\omega : X \neq Y\} \subset \cup_{n\in\mathbb{N}} \{\omega : |X - Y| > 2/n\} = \cup_{n\in\mathbb{N}} \{\omega : |X - Y| > 2/n\},$$

which is a null set, completing the proof.