Econ 7818, Homework 4, Professor Martins. Due date: at the time of the final examination.

1. Let $X, Y \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ be random variables and assume that $E(Y|X) = aX$ where $a \in \mathbb{R}$.
   (a) Show that if $E(X^2) > 0$, $a = E(XY)/E(X^2)$.
   (b) If $\{(Y_i\ X_i)^T\}_{i=1}^n$ is a sequence of independent random vectors with components having the same distribution as $(Y\ X)^T$, show that
   $$\frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{p} E(X^2) \text{ and } \frac{1}{n}\sum_{i=1}^n Y_i X_i \xrightarrow{p} E(XY).$$
   (c) Let $a_n = \left(\frac{1}{n}\sum_{i=1}^n X_i^2\right)^{-1} \frac{1}{n}\sum_{i=1}^n Y_i X_i$. Does $a_n \xrightarrow{p} a$? Can a_n be defined for all n? Explain.
   (d) Show that $\sqrt{n}(a_n - a) \xrightarrow{d} Z \sim N(0, V)$. What is V?

2. Prove the following:
   (a) If $Y \in \mathcal{L}(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra, show that $|E(Y|\mathcal{G})| \leq E(|Y||\mathcal{G})$.
   (b) Let c be a scalar constant and suppose $X = c$ almost surely. Show that $E(X|\mathcal{G}) = c$ almost surely.
   (c) If $Y \in \mathcal{L}(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra, show that for $a > 0$
   $$P(\{\omega : |Y(\omega)| \geq a\}|\mathcal{G}) \leq \frac{1}{a} E(|Y(\omega)||\mathcal{G}).$$
   What is the definition of $P(\{\omega : |Y(\omega)| \geq a\}|\mathcal{G})$? Is this a legitimate probability measure?

3. Let Y and X be random variables such that $Y, X \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ and define $\varepsilon = Y - E(Y|X)$.
   (a) Show that $E(\varepsilon|X) = 0$ and $E(\varepsilon) = 0$.
   (b) Let $V(Y|X) = E(Y^2|X) - E(Y|X)^2$. Show that $V(Y|X) = V(\varepsilon|X)$, $V(\varepsilon) = E(V(Y|X))$;
   (c) $Cov(\varepsilon, h(X)) = 0$ for any function of X whose expectation exists.
   (d) Assume that $E(Y|X) = \alpha + \beta X$ where $\alpha, \beta \in \mathbb{R}$. Let $E(Y) = \mu_Y$, $E(X) = \mu_X$, $V(Y) = \sigma_Y^2$, $V(X) = \sigma_X^2$ and $\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$. Show that,
   $$E(Y|X) = \mu_Y + \rho\sigma_Y \frac{X - \mu_X}{\sigma_X} \text{ and } E(V(Y|X)) = (1 - \rho^2)\sigma_Y^2.$$

4. Suppose $\{X_i\}_{i=1,2,\ldots}$ is a sequence of independent and identically distributed random variables and $Y_i(x) = I_{\{\omega : X_i \leq x\}}$, where I_A is the indicator function of the set A. Now define
   $$F_n(x) = \frac{1}{n}\sum_{i=1}^n Y_i(x)$$
   for fixed x. Obtain the asymptotic distribution of $\sqrt{n}(F_n(x) - F(x))$. You can use a Central Limit Theorem, but otherwise show all your work.

5. Let $\{X_n\}_{n=1,2,\ldots}$ and $\{Y_n\}_{n=1,2,\ldots}$ be sequences of random variables defined on the same probability space. Suppose $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ and assume X_n and Y_n are independent for all n and X and Y are independent. Show that $X_n + Y_n \xrightarrow{d} X + Y$. Hint: use the characteristic function for a sum of independent random variables.

6. Let $\{X_i\}_{i=1,2,\cdots}$ be a sequence of independent and identically random variables with $E(X_i) = 1$ and $\sigma^2_{X_i} = \sigma^2 < \infty$. Show that if $S_n = \sum_{i=1}^{n} X_i$

$$\frac{2}{\sigma}\left(S_n^{1/2} - n^{1/2}\right) \xrightarrow{d} Z \sim N(0,1).$$