Econ 7818, Homework 4, Professor Martins. Due date: at the time of the final examination.

1. Suppose $X_n \xrightarrow{d} X$ where X has a continuous distribution. Show that in this case $\sup_{x\in\mathbb{R}}|P(X_n \leq x) - P(X \leq x)| \to 0$. That is, convergence in distribution is uniform in $\mathbb{R}$.

2. Let $(\Omega, \mathcal{F}, P)$ be a probability space. The set of random variables $X : \Omega \to \mathbb{R}$ such that $\int_\Omega X^2 dP < \infty$ is denoted by $L^2(\Omega, \mathcal{F}, P)$. On this set $\|X\| = \left(\int_\Omega X^2 dP\right)^{1/2}$ is a norm and $< X, Y >= \int_\Omega XY dP$ is an inner product. If $\mathcal{G}$ is a σ-algebra and $\mathcal{G} \subset \mathcal{F}$, the conditional expectation of X with respect to $\mathcal{G}$, denoted by $E(X|\mathcal{G})$ is the orthogonal projection of X onto the closed subspace $L^2(\Omega, \mathcal{G}, P)$ of $L^2(\Omega, \mathcal{F}, P)$. Prove the following results:

   (a) For $X, Y \in L^2(\Omega, \mathcal{F}, P)$ we have $< E(X|\mathcal{G}), Y >=< E(Y|\mathcal{G})), X >=< E(X|\mathcal{G}), E(Y|\mathcal{G}) >$.

   (b) If $X = Y$ almost everywhere then $E(X|\mathcal{G}) = E(Y|\mathcal{G})$ almost everywhere.

   (c) For $X \in L^2(\Omega, \mathcal{G}, P)$ we have $E(X|\mathcal{G}) = X$.

   (d) If $\mathcal{H} \subset \mathcal{G}$ is a σ-algebra, then $E(E(X|\mathcal{G})|\mathcal{H}) = E(X|\mathcal{H})$.

   (e) If $Y \in L^2(\Omega, \mathcal{G}, P)$ and there exists a constant $C > 0$ such that $P(|Y| \geq C) = 0$, we have that $E(YX|\mathcal{G}) = YE(X|\mathcal{G})$.

   (f) If $\{Y_n\}_{n\in\mathbb{N}}$, $X \in L^2(\Omega, \mathcal{F}, P)$ and $\|Y_n - X\| \to 0$ as $n \to \infty$, then $E(Y_n|\mathcal{G}) \xrightarrow{p} E(X|\mathcal{G})$ as $n \to \infty$.

3. Let $X, Y \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ be random variables and assume that $E(Y|X) = aX$ where $a \in \mathbb{R}$.

   (a) Show that if $E(X^2) > 0$, $a = E(XY)/E(X^2)$.

   (b) If $\{(Y_i \ X_i)^T\}_{i=1}^n$ is a sequence of independent random vectors with components having the same distribution as $(Y \ X)^T$, show that

   $$\frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{p} E(X^2) \text{ and } \frac{1}{n}\sum_{i=1}^n Y_i X_i \xrightarrow{p} E(XY).$$

   (c) Let $a_n = \left(\frac{1}{n}\sum_{i=1}^n X_i^2\right)^{-1} \frac{1}{n}\sum_{i=1}^n Y_i X_i$. Does $a_n \xrightarrow{p} a$? Can a_n be defined for all n? Explain.

   (d) Show that $\sqrt{n}(a_n - a) \xrightarrow{d} Z \sim N(0, V)$. What is V?

4. Prove the following:

   (a) If $Y \in \mathcal{L}(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra, show that $|E(Y|\mathcal{G})| \leq E(|Y||\mathcal{G})$.

   (b) Let c be a scalar constant and suppose $X = c$ almost surely. Show that $E(X|\mathcal{G}) = c$ almost surely.

   (c) If $Y \in \mathcal{L}(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra, show that for $a > 0$

   $$P\left(\{\omega : |Y(\omega)| \geq a\}|\mathcal{G}\right) \leq \frac{1}{a}E(|Y(\omega)||\mathcal{G}).$$

   What is the definition of $P\left(\{\omega : |Y(\omega)| \geq a\}|\mathcal{G}\right)$? Is this a legitimate probability measure?

5. Let Y and X be random variables such that $Y, X \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ and define $\varepsilon = Y - E(Y|X)$.

   (a) Show that $E(\varepsilon|X) = 0$ and $E(\varepsilon) = 0$.

   (b) Let $V(Y|X) = E(Y^2|X) - E(Y|X)^2$. Show that $V(Y|X) = V(\varepsilon|X)$, $V(\varepsilon) = E(V(Y|X))$;

   (c) $Cov(\varepsilon, h(X)) = 0$ for any function of X whose expectation exists.

(d) Assume that $E(Y|X) = \alpha + \beta X$ where $\alpha, \beta \in \mathbb{R}$. Let $E(Y) = \mu_Y$, $E(X) = \mu_X$, $V(Y) = \sigma_Y^2$, $V(X) = \sigma_X^2$ and $\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$. Show that,

$$E(Y|X) = \mu_Y + \rho\sigma_Y \frac{X - \mu_X}{\sigma_X} \text{ and } E(V(Y|X)) = (1 - \rho^2)\sigma_Y^2.$$

6. Let $\{X_n\}_{n=1,2,\cdots}$ and $\{Y_n\}_{n=1,2,\cdots}$ be sequences of random variables defined on the same probability space. Suppose $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ and assume X_n and Y_n are independent for all n and X and Y are independent. Show that $X_n + Y_n \xrightarrow{d} X + Y$. Hint: use the characteristic function for a sum of independent random variables.

7. Let $\{X_i\}_{i=1,2,\cdots}$ be a sequence of independent and identically random variables with $E(X_i) = 1$ and $\sigma_{X_i}^2 = \sigma^2 < \infty$. Show that if $S_n = \sum_{i=1}^n X_i$

$$\frac{2}{\sigma}\left(S_n^{1/2} - n^{1/2}\right) \xrightarrow{d} Z \sim N(0,1).$$