Econ 7818 - Mathematical Statistics, Professor Martins.
Midterm Examination
Date: October 20, 2022
Exam duration: From 9:30 AM to 10:50 AM. An additional 5 minutes will be allowed, if needed.

Important instructions: The pages with your answers should be numbered consecutively starting at 1, i.e., $\{1, 2, \dots m\}$, $m \in \mathbb{N}$. Write only on one-side of each sheet of paper. Start the answer for a new question on a new sheet of paper. No books, notes, computers, tablets, etc. allowed during the exam.

Question 1: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_i : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B})$ for $i = 1, 2, \cdots, n$ and $n \in \mathbb{N}$ be random variables.

1. Let $\delta_x : \mathcal{B} \to \mathbb{R}$ be a set function defined as $\delta_x(B) = \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{if } x \notin B \end{cases}$. Show that δ_x is a probability measure on $(\mathbb{R}, \mathcal{B})$.

   Answer: We must show that δ_x is a probability measure on $(\mathbb{R}, \mathcal{B})$. By definition of δ_x we have $\delta_x(\emptyset) = 0$, $\delta_x(\mathbb{R}) = 1$. Now, if $\{A_n\}_{n \in \mathbb{N}}$ is a pairwise disjoint sequence of sets,

   $$\delta_x\left(\cup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \delta_x(A_n)$$

   establishing σ-additivity.

2. Let $P_n : \mathcal{B} \to \mathbb{R}$ be a set function defined as $P_n(B) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}(B)$. Show that P_n is a probability measure on $(\mathbb{R}, \mathcal{B})$.

   Answer: From item 1, $P_n(\emptyset) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}(\emptyset) = 0$ and $P_n(\mathbb{R}) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}(\mathbb{R}) = \frac{1}{n} n = 1$. Now, if $\{A_n\}_{n \in \mathbb{N}}$ is a pairwise disjoint sequence of sets,

   $$P_n\left(\cup_{n \in \mathbb{N}} A_n\right) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}(\cup_{n \in \mathbb{N}} A_n) = \frac{1}{n} \sum_{i=1}^{n} \sum_{n \in \mathbb{N}} \delta_{X_i}(A_n).$$

   Now, if $\beta_{ik} := \sum_{n=1}^{k} \delta_{X_i}(A_n)$ for $k = 1, 2, \cdots$ we note that $\beta_{i1} \leq \beta_{i2} \leq \cdots$ and $0 \leq \beta_{ik} \leq 1$. Hence, $\sum_{j=1}^{\infty} \delta_{X_i}(A_n) = \lim_{N \to \infty} \beta_{iN}$. Then,

   $$\begin{aligned} P_n\left(\cup_{n \in \mathbb{N}} A_n\right) &= \frac{1}{n} \sum_{i=1}^{n} \lim_{N \to \infty} \beta_{iN} = \lim_{N \to \infty} \frac{1}{n} \sum_{i=1}^{n} \beta_{iN} \\ &= \lim_{N \to \infty} \frac{1}{n} \sum_{i=1}^{n} \sum_{n=1}^{N} \delta_{X_i}(A_n) = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}(A_n) = \sum_{n=1}^{\infty} P_n(A_n) \end{aligned}$$

3. Let $g : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$ be a measurable nonnegative simple function. Show that $\int_{\mathbb{R}} g d\delta_x = g(x)$ and use this result to show that for any $f : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$ measurable $\int_{\mathbb{R}} f d\delta_x = f(x)$.

   Answer: If g is simple, $g(x) = \sum_{j=0}^{M} a_j I_{A_j}(x)$, where M is finite and $\{A_j\}$ is a disjoint collection with $\cup_{j \in \mathbb{N}} A_j = \mathbb{R}$. Let $x \in \mathbb{R}$, then x is in a single A_j, say $A_{j'}$. Then

   $$\int g d\delta_x = \int \sum_{j=0}^{M} a_j I_{A_j} d\delta_X = \sum_{j=0}^{M} a_j \int I_{A_j} d\delta_X = \sum_{j=0}^{M} a_j \delta_X(A_j) = a_j' = g(x).$$

   For every positive f, we have that $f(x) = \lim_{n \to \infty} f_n(x)$ with $0 \leq f_1 \leq f_2 \leq \cdots$ where f_n is simple.

   $$\int f d\delta_x = \int \lim f_n d\delta_x = \lim \int f_n d\delta_x = \lim f_n(x) = f(x).$$

For arbitrary measurable f, we note that $f = f^+ - f^-$. Hence, $\int f d\delta_x = \int f^+ d\delta_x - \int f^- d\delta_x = f^+(x) - f^-(x) = f(x)$.

4. Show that for $f : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$ measurable, $\int_{\mathbb{R}} f dP_n = \frac{1}{n} \sum_{i=1}^n f(X_i)$.

   Answer: For g simple

$$\int g dP_n = \sum_{j=0}^M a_j \int I_{A_j} dP_n = \sum_{j=0}^M a_j P_n(A_j) = \sum_{j=0}^M a_j \frac{1}{n} \sum_{i=1}^n \delta_{X_i}(A_j)$$
$$= \frac{1}{n} \sum_{i=1}^n \sum_{j=0}^M a_j \delta_{X_i}(A_j) = \frac{1}{n} \sum_{i=1}^n g(X_i).$$

   For f positive,

$$\int f dP_n = \sum_{j=0}^M a_j \int I_{A_j} dP_n = \sum_{j=0}^M a_j P_n(A_j) = \sum_{j=0}^M a_j \frac{1}{n} \sum_{i=1}^n \delta_{X_i}(A_j)$$
$$= \frac{1}{n} \sum_{i=1}^n \sum_{j=0}^M a_j \delta_{X_i}(A_j) = \frac{1}{n} \sum_{i=1}^n g(X_i).$$

   For arbitrary measurable f, we write $f = f^+ - f^-$ and

$$\int f dP_n = \int f^+ dP_n - \int f^- dPn = \frac{1}{n} \sum_{i=1}^n f^+(X_i) - \frac{1}{n} \sum_{i=1}^n f^-(X_i) = \frac{1}{n} \sum_{i=1}^n (f^+(X_i) - f^-(X_i))$$
$$= \frac{1}{n} \sum_{i=1}^n f(X_i)$$

Question 2: Prove the following:

1. Show that if $V : (\Omega, \mathcal{F}, P) \to (\mathbb{R}^p, \mathcal{B}^p)$ for $p \in \mathbb{N}$ is a random vector, then the σ-algebra generated by any of its components is a subset of the σ-algebra generated by V.

   Answer: Since V is a random vector $V^{-1}(\mathcal{B}^p) := \sigma(V) \subset \mathcal{F}$. Let $g : \mathbb{R}^p \to \mathbb{R}$ be such that $g(x) = x_i$, where x_i is the ith component of x. It is easy to verify that g is continuous and therefore measurable. Hence, $X_i^{-1}(B) \in \mathcal{B}^p$ for all $B \in \mathcal{B}$. In addition, $V^{-1}(X_i^{-1}(B)) \in \sigma(V)$. Hence, $\sigma(X_i) = X_i^{-1}(\mathcal{B}) \subseteq \sigma(V)$.

2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $E_s, E \in \mathcal{F}$ with $E_s \subset E$ and $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B})$ be a random variable. Show that for every $\epsilon > 0$ there is some $\delta > 0$ such that

$$P(E - E_s) < \delta \implies \left| \int_E X dP - \int_{E_s} X dP \right| < \epsilon.$$

   Answer: Let $A = E - E_s$. Since $E, E_s \in \mathcal{F}$, we have $A \in \mathcal{F}$. Then, we must show that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$P(A) < \delta \implies | \int_E X dP - \int_{E_s} X dP | = | \int_{E - E_s} X dP | = | \int_A X dP | \leq \int_A |X| dP < \epsilon.$$

   Let s_n be simple functions. Then, $|s_n|$ are simple functions and we can write $\lim_{n\to\infty} |s_n| = |X|$ with $|s_1| \leq |s_2| \leq \cdots \leq |X|$. Also, given that $X = X^+ - X^-$ we conclude that $\lim_{n\to\infty} s_n = X$. Since X is integrable, by Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty} \int |s_n - X| dP = 0.$$

That is, for $n > N_\epsilon$ we have $\int |s_n - X| dP < \epsilon/2$. Then, since $|s_n|$ is bounded by the definition of simple functions,

$$\int_A |X| dP = \int_A |X - s_n + s_n| dP \leq \int_A |X - s_n| dP + \int_A |s_n| dP \leq \epsilon/2 + CP(A) \leq \epsilon/2 + C\delta.$$

Letting $C = \epsilon/2\delta$ gives the desired result.

$$\begin{aligned}
\int_A |X| dP &= \int_{A\cap\{\omega:|X(\omega)|\geq C\}} |X| dP + \int_{A\cap\{\omega:|X(\omega)|<C\}} |X| dP \\
&\leq \int_{A\cap\{\omega:|X(\omega)|\geq C\}} |X| dP + CP(\{A \cap \{\omega : |X(\omega)| < C\}) \leq \int_{A\cap\{\omega:|X(\omega)|\geq C\}} |X| dP + CP(A) \\
&\leq \int_{A\cap\{\omega:|X(\omega)|\geq C\}} |X| dP + \epsilon \text{ by choosing } \delta = \epsilon/C \text{ since } P(A) < \delta.
\end{aligned}$$