Econ 7818 - - Introduction to Probability and Asymptotic Theory, Professor Martins.
Midterm Examination
Date: October 19, 2023
Exam duration: From 9:30 AM to 10:50 AM. An additional 5 minutes will be allowed, if needed.

Important instructions: The pages with your answers should be numbered consecutively starting at 1, i.e., $\{1, 2, \dots m\}$, $m \in \mathbb{N}$. Write only on one-side of each sheet of paper. Start the answer for a new question on a new sheet of paper. No books, notes, computers, tablets, etc. allowed during the exam.

Question 1: Consider the following:

1. Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables, and X be a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$. Suppose that $E(X_n) < \infty$ and that $X_n \to X$ uniformly in Ω, i.e., for all $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ (depending only on ϵ, not ω) such that for $n \geq N(\epsilon)$

$$|X_n(\omega) - X(\omega)| < \epsilon \text{ for all } \omega \in \Omega.$$

First, show that $E(X) < \infty$. Second, show that $\lim_{n\to\infty} E(X_n) = E\left(\lim_{n\to\infty} X_n\right) = E(X)$. Why is uniform convergence of $\{X_n\}_{n\in\mathbb{N}}$ needed? Wouldn't convergence on Ω be enough? Explain.

Answer: First, we need to show that $E(X) < \infty$. Now, $E(X) < \infty \iff \int_\Omega |X| dP < \infty$. Hence, choose any $\epsilon > 0$ such that $n \geq N(\epsilon)$ gives $|X_n(\omega) - X(\omega)| < \epsilon$ for all $\omega \in \Omega$. Then,

$$\begin{aligned}
\int_\Omega |X| dP &= \int_\Omega |X - X_n + X_n| dP \leq \int_\Omega (|X - X_n| + |X_n|)\, dP \\
&\leq \int_\Omega (\epsilon + |X_n|)\, dP \text{ by uniform convergence} \\
&= \epsilon + \int_\Omega |X_n| dP < \infty \text{ by the fact that } P(\Omega) = 1 \text{ and } E(X_n) < \infty.
\end{aligned}$$

Now, to show that $\lim_{n\to\infty} E(X_n) = E(X)$, consider

$$|E(X_n) - E(X)| = \left| \int_\Omega X_n dP - \int_\Omega X dP \right| = \left| \int_\Omega (X_n - X) dP \right| \leq \int_\Omega |X_n - X| dP \leq \epsilon$$

by uniform convergence. Uniform convergence is needed because we need to bound $|X_n(\omega) - X(\omega)|$ for all ω *simultaneously*.

2. Now, let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real valued measurable functions, and f be a real valued measurable function defined on $(\mathbb{R}, \mathcal{B}, \lambda)$. Suppose that $\int_\mathbb{R} f_n d\lambda$, $\int_\mathbb{R} f d\lambda < \infty$ and that $f_n \to f$ uniformly on $\mathbb{R}$. Is $\lim_{n\to\infty} \int_\mathbb{R} f_n d\lambda = \int_\mathbb{R} f d\lambda$? Why, or why not? Hint: Think of the sequence $f_n(x) = \frac{1}{2n} I_{[-n,n]}$.
Does the result in item 1 contradict your answer for item 2? Why? Why not?

Answer: No, the limit is incorrect. Take the hint and note that

$$\int_\mathbb{R} f_n d\lambda = \frac{1}{2n} \int_\mathbb{R} I_{[-n,n]} d\lambda = \frac{1}{2n} \lambda([-n, n]) = 1.$$

Hence, $\lim_{n\to\infty} \int_\mathbb{R} f_n d\lambda = 1$. Also, note that for any $\epsilon > 0$

$$|f_n(x) - 0| = |\frac{1}{2n} I_{[-n,n]}| \leq \frac{1}{2n} < \epsilon$$

whenever $n > N(\epsilon) > \frac{1}{2\epsilon}$. Hence, f_n converges uniformly to $f = 0$, and $\int_{\mathbb{R}} f d\lambda = 0 \neq \lim \int_{\mathbb{R}} f_n d\lambda = 1$.

Question 2: Prove the following:

1. If $V : (\Omega, \mathcal{F}, P) \to (\mathbb{R}^p, \mathcal{B}^p)$ for $p \in \mathbb{N}$ is a random vector, then the σ-algebra generated by any of its components is a subset of the σ-algebra generated by V.

   Answer: Since V is a random vector $V^{-1}(\mathcal{B}^p) := \sigma(V) \subset \mathcal{F}$. Let $g : \mathbb{R}^p \to \mathbb{R}$ be such that $g(x) = x_i$, where x_i is the ith component of x. It is easy to verify that g is continuous and therefore measurable. Hence, $V_i^{-1}(B) \in \mathcal{B}^p$ for all $B \in \mathcal{B}$. In addition, $V^{-1}(V_i^{-1}(B)) \in \sigma(V)$. Hence, $\sigma(V_i) = V_i^{-1}(\mathcal{B}) \subset \sigma(V)$.

2. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $E_s, E \in \mathcal{F}$ with $E_s \subset E$ and $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B})$ be a random variable. Show that for every $\epsilon > 0$ there is some $\delta > 0$ such that

$$P(E - E_s) < \delta \implies \left| \int_E X dP - \int_{E_s} X dP \right| < \epsilon.$$

   Answer: Let $A = E - E_s$. Since $E, E_s \in \mathcal{F}$, we have $A \in \mathcal{F}$. Since

$$\left| \int_E X dP - \int_{E_s} X dP \right| = \left| \int_{E - E_s} X dP \right| = \left| \int_A X dP \right| \leq \int_A |X| dP,$$

   it suffices to show that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$P(A) < \delta \implies \int_A |X| dP < \epsilon.$$

   Now, $X = X^+ - X^-$ and X^+ and X^- are non-negative random variables. Hence, there exist increasing sequences of non-negative measurable functions $\{g_n\}_{n \in \mathbb{N}}$ and $\{h_n\}_{n \in \mathbb{N}}$ such that $g_n \to X^+$ and $h_n \to X^-$ as $n \to \infty$. Hence, $s_n := g_n - h_n \to X$ as $n \to \infty$ and

$$|s_n| \leq |g_n| + |h_n| \leq X^+ + X^- = |X|.$$

   Hence, if X is integrable, i.e., $\int |X| dP < \infty$, by Lebesgue's dominated convergence theorem,

$$\lim_{n \to \infty} \int |s_n - X| dP = 0.$$

   That is, for $n > N_\epsilon$ we have $\int |s_n - X| dP < \epsilon/2$. Then, since $|s_n|$ is bounded by the definition of simple functions,

$$\int_A |X| dP = \int_A |X - s_n + s_n| dP \leq \int_A |X - s_n| dP + \int_A |s_n| dP \leq \epsilon/2 + CP(A) \leq \epsilon/2 + C\delta.$$

   Letting $C = \epsilon/2\delta$ gives the desired result.