Econ 7818 - Introduction to Probability and Asymptotic Theory, Professor Martins.
Midterm Examination
Date: October 14, 2025
Exam duration: From 9:30 AM to 10:55 AM.

Important instructions: The pages with your answers should be numbered consecutively starting at 1, i.e., $\{1, 2, \ldots m\}$, $m \in \mathbb{N}$. Write only on one-side of each sheet of paper. Start the answer for a new question on a new sheet of paper. No books, notes, computers, tablets, etc. allowed during the exam.

Question 1: Let $(\mathbb{X}, \mathcal{F}, \mu)$ be a measure space and $\mathcal{E} \subset \mathcal{F}$ be a σ-algebra. Denote $\nu := \mu|_{\mathcal{E}}$ the restriction of μ to $\mathcal{E}$.

1. Show that ν is a measure,
2. Assume that μ is a probability measure. Is ν a probability measure?
3. If μ is σ-finite, is ν σ-finite? If so, prove. If not, give an example where μ is σ-finite, but ν isn't.

Hint: A measure μ is σ-finite and $(\mathbb{X}, \mathcal{F}, \mu)$ is called a σ-finite measure space if $\mathcal{F}$ contains a sequence $\{F_n\}_{n \in \mathbb{N}}$ such that $F_n \uparrow \mathbb{X}$ and $\mu(F_n) < \infty$.

Answer: 1. is obvious since $\emptyset \in \mathcal{E} \subset \mathcal{F}$ and a disjoint collection $\{E_i\}_{i \in \mathbb{N}} \subset \mathcal{E} \subset \mathcal{F}$. Hence, ν inherits the properties of μ.

2. Since, $\mathbb{X} \in \mathcal{E}$ and $\mathbb{X} \in \mathcal{F}$ we immediately get $\nu(\mathbb{X}) = \mu(\mathbb{X}) = 1$.

3. No. σ-finiteness is a property of measures **and** σ-algebras. Here is an example: Let $(\mathbb{R}, \mathcal{B}, \lambda)$ be a measure space and $\mathcal{E} = \{\mathbb{R}, (-\infty, 0), [0, \infty), \emptyset\}$. $(\mathbb{R}, \mathcal{B}, \lambda)$ is σ-finite, but $(\mathbb{R}, \mathcal{E}, \lambda)$ isn't since it does not contain an exhausting sequence with components that have finite measure.

Question 2: Let $(\Omega, \mathcal{F}, P)$ be a probability space and suppose that $\mathcal{F}$ admits a *filtration*, i.e., there exists an increasing sequence of σ-algebras such that $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}$. Now, let $X_n : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B})$ for $n \in \mathbb{N}_0 = \{0, 1, \cdots\}$ be random variables. The sequence $\{X_n\}_{n \in \mathbb{N}_0}$ is called a martingale if X_n is $\mathcal{F}_n - \mathcal{B}$ measurable, $X_n \in \mathcal{L}(\Omega, \mathcal{F}_n, P)$ for each n and

$$\int_F X_{n+1} dP = \int_F X_n dP \text{ for all } F \in \mathcal{F}_n. \tag{1}$$

1. Prove that we can replace $\mathcal{F}_n$ in equation (1) with a π system that generates $\mathcal{F}_n$ containing and exhausting sequence.
2. If $\mathcal{F}_0 := \{\emptyset, \Omega\}$, show that $\{X_n\}_{n \in \mathbb{N}_0}$ is a martingale if, and only if, $X_0 = E(X_1)$.

Hint: For item 1, use Theorems 2.4 and 4.11 from your class notes. They are stated below:

Theorem 2.4: *Let $(\mathbb{X}, \sigma(\mathcal{P}))$ be a measurable space and $\mathcal{P}$ a collection of subsets of $\mathbb{X}$, such that:*

1. *$\mathcal{P}$ is a π system,*
2. *there exists $\{P_j\}_{j \in \mathbb{N}} \subset \mathcal{P}$ with $P_1 \subset P_2 \subset \cdots$ such that $\bigcup_{j \in \mathbb{N}} P_j := \lim_{j \to \infty} P_j = \mathbb{X}$ (the sequence $\{P_j\}_{j \in \mathbb{N}}$ is exhausting).*

If μ and v are measures that coincide on $\mathcal{P}$ and, are finite for all P_j, then $\mu(A) = v(A)$ for all $A \in \sigma(\mathcal{P})$.

Theorem 4.11: *Let $f : (\mathbb{X}, \mathcal{F}, \mu) \to (\bar{\mathbb{R}}, \bar{\mathcal{B}})$ be a a non-negative measurable function such that $f \in \mathcal{L}_{\bar{\mathcal{B}}}$ and $m(E) = \int_E f d\mu$ for all $E \in \mathcal{F}$. Then, m is a measure on $\mathcal{F}$.*

Answer: 1. Recall that $X_n = X_n^+ - X_n^-$. Hence, equation (1) can be written as

$$\int_F (X_{n+1}^+ - X_{n+1}^-) dP = \int_F (X_n^+ - X_n^-) dP \iff \int_F (X_{n+1}^+ + X_n^-) dP = \int_F (X_{n+1}^- + X_n^+) dP,$$

where $X_{n+1}^+ + X_n^- \geq 0$ and $X_{n+1}^- + X_n^+ \geq 0$. In addition, given integrability of X_n and X_{n+1}, the integrals on both sides of the last equality are finite (real valued). Hence, by Theorem 4.11 we can define

$$\mu(F) := \int_F (X_{n+1}^+ + X_n^-) dP \text{ and } v(F) = \int_F (X_{n+1}^- + X_n^+) dP$$

and conclude that they are finite measures on $\mathcal{F}_n$. Hence, the martingale condition is equivalent to a statement about equality of measures defined on $\mathcal{F}_n$. Now, by Theorem 2.4, if there is a π-system $\mathcal{P}$, containing an exhausting sequence, that generates $\mathcal{F}_n$ then if μ and v coincide on the π-system they will coincide on $\mathcal{F}_n$. If that is the case, equation (1) will hold if it holds for the sets in the π-system.

2. X_0 must be $\mathcal{F}_0 - \mathcal{B}$ measurable. That is, for every $B \in \mathcal{B}$, $X_0^{-1}(B) = \Omega$ or $X_0^{-1}(B) = \emptyset$. Hence, X_0 must be a constant. Now,

$$\int_\Omega X_0 dP = X_0 \int_\Omega dP = X_0 P(\Omega) = X_0 = \int_\Omega X_1 dP = E(X_1),$$

where the next to last equality follows from equation (1).

Now, conversely, since $\int_\emptyset X_0 dP = \int_\emptyset X_1 dP$ always holds and we have already verified that $\int_\Omega X_0 dP = \int_\Omega X_1 dP$. Hence, since $\mathcal{F}_0 = (\Omega, \emptyset)$, there is no other choice for X_0.

Question 3: We used Beppo Levi's Theorem repeatedly in class. In particular, it was crucial to prove Fatou's Lemma. Use it again to prove the following more general version of Fatou's Lemma.

Generalized Fatou's Lemma: *Assume that $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of integrable functions defined on a measure space with measure μ. Then,*

1. *if $f_n \geq v$ for all n and some integrable v then $\int \liminf f_n d\mu \leq \liminf \int f_n d\mu$,*
2. *if $f_n \leq w$ for all n and some integrable w then $\limsup \int f_n d\mu \leq \int \limsup f_n d\mu$.*

Answer: For 1., $f_n - v \geq 0$ for all n. Since v is integrable $f_n - v$ is integrable. Then, by Fatou's Lemma

$$\int \liminf (f_n - v) d\mu \leq \liminf \int (f_n - v) d\mu \iff \int \liminf f_n d\mu - \int v d\mu \leq \liminf \int f_n d\mu - \int v d\mu$$

which gives $\int \liminf f_n d\mu \leq \liminf \int f_n d\mu$.

For 2., $w - f_n \geq 0$ for all n. Since w is integrable $w - f_n$ is integrable. Then, by Fatou's Lemma,

$$\int \liminf (w - f_n) d\mu \leq \liminf \int (w - f_n) d\mu \iff \int w d\mu - \int \limsup f_n d\mu \leq \int w d\mu - \limsup \int f_n d\mu$$

since $\liminf(-f_n) = -\limsup f_n$. Hence, $\int \limsup f_n d\mu \geq \limsup \int f_n d\mu$.