

Important instructions: The pages with your answers should be numbered consecutively starting at 1, i.e., $\{1, 2, \dots, m\}$, $m \in \mathbb{N}$. Write only on one-side of each sheet of paper. Start the answer for a new question on a new sheet of paper. No books, notes, computers, tablets, etc. allowed during the exam.

Question 1: This question deals with the concept of measurable sets and functions, as well as certain properties of probability measures.

1. Let $(\mathbb{X}, \mathcal{F}, \mu)$ be a measure space and let $f : (\mathbb{X}, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be measurable. Suppose $g : \mathbb{X} \rightarrow \mathbb{R}$ satisfies $g = f$ μ -a.e.

(a) Show that if $(\mathbb{X}, \mathcal{F}, \mu)$ is *complete*, then g is measurable.

- (b) Let $\mathbb{X} = \{1, 2, 3\}$, $\mathcal{F} = \{\emptyset, \{1\}, \{2, 3\}, \mathbb{X}\}$, $\mu(\{1\}) = 1$, $\mu(\{2, 3\}) = 0$, $f := 0$ and $g = I_{\{2\}}$. Is f measurable? Is $f = g$ a.e.? Is g measurable? Is $(\mathbb{X}, \mathcal{F}, \mu)$ complete?

Answer: (a) Let $N := \{x : f(x) \neq g(x)\}$. By hypothesis N is contained in a μ -null set, so completeness (Definition 1.9) gives $N \in \mathcal{F}$, $\mu(N) = 0$, and *every* subset of N lies in $\mathcal{F}$. For $a \in \mathbb{R}$,

$$\{g > a\} = (\{g > a\} \cap N^c) \cup (\{g > a\} \cap N) = (\{f > a\} \cap N^c) \cup (\{g > a\} \cap N).$$

The first set is in $\mathcal{F}$ because f is measurable; the second is a subset of N , hence in $\mathcal{F}$ by completeness. So $\{g > a\} \in \mathcal{F}$ for every a , and by the characterization on pp. 59–60, g is $\mathcal{F}$ - $\mathcal{B}(\mathbb{R})$ measurable.

- (b) Yes, f is measurable. $\{f \neq g\} = \{2\} \subset \{2, 3\}$, a null set, so $g = f$ a.e.; but $\{g > 1/2\} = \{2\} \notin \mathcal{F}$, so g is not measurable. The space is not complete precisely because $\{2\} \subset \{2, 3\} \in \mathcal{N}_\mu$ is not in $\mathcal{F}$.

2. Show that a probability measure P on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying $P(A+t) = P(A)$ for every Borel set A and every $t \in \mathbb{R}$ (translation invariance) cannot exist.

Answer: Put $c = P([0, 1))$. The disjoint sets $[k, k+1)$, $k \in \mathbb{Z}$, partition $\mathbb{R}$ and each has probability c . Countable additivity yields $1 = \sum_{k \in \mathbb{Z}} c$, which is 0 if $c = 0$ and $+\infty$ if $c > 0$. Both are impossible.

3. Let $f_n : (\mathbb{X}, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be measurable for $n \in \mathbb{N}$. Show that

$$A := \left\{x \in \mathbb{X} : \lim_{n \rightarrow \infty} f_n(x) \text{ exists and is finite}\right\} \in \mathcal{F}.$$

Hint: Show that $A = \bigcap_{k \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{m, n \geq N} \left\{x : |f_n(x) - f_m(x)| < \frac{1}{k}\right\}$ using the fact that every Cauchy sequence in $\mathbb{R}$ converges to a limit in $\mathbb{R}$.

Answer: First show that $A = B$, where $B := \bigcap_{k \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{m, n \geq N} \left\{x : |f_n(x) - f_m(x)| < \frac{1}{k}\right\}$. Intersections correspond to “for all” and unions to “there exists”. Hence

$$x \in B \iff \forall k \in \mathbb{N} \exists N \in \mathbb{N} \forall m, n \geq N : |f_n(x) - f_m(x)| < \frac{1}{k}. \quad (1)$$

Let $x \in A$ and set $L := \lim_{n \rightarrow \infty} f_n(x) \in \mathbb{R}$. Fix $k \in \mathbb{N}$. By convergence there exists $N \in \mathbb{N}$ such that $|f_n(x) - L| < \frac{1}{2k}$ for all $n \geq N$. Then for all $m, n \geq N$,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - L| + |L - f_m(x)| < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k}.$$

Since k was arbitrary, (1) holds, so $x \in B$.

Let $x \in B$ and write $a_n := f_n(x)$. (a_n) is Cauchy. Let $\varepsilon > 0$. By the Archimedean property there is $k \in \mathbb{N}$ with $\frac{1}{k} < \varepsilon$. By (1) there is $N \in \mathbb{N}$ with $|a_n - a_m| < \frac{1}{k} < \varepsilon$ for all $m, n \geq N$. A Cauchy sequence in $\mathbb{R}$ converges to a finite limit in $\mathbb{R}$. Thus $\lim_{n \rightarrow \infty} f_n(x) = L$ exists and is finite, i.e. $x \in A$.

Now, for each m, n the function $f_n - f_m$ is measurable, and so is $|f_n - f_m|$, by Theorem 3.5; hence each set $\{|f_n - f_m| < 1/k\} \in \mathcal{F}$. The displayed expression involves only countable intersections and unions, so $A \in \mathcal{F}$.

Question 2: This question deals with uses of convergence theorems and L^p spaces.

1. If X is an integrable random variable, show that

$$\lim_{t \rightarrow \infty} tP(|X| > t) = 0.$$

Hint: Use Lebesgue's Dominated Convergence Theorem.

Answer: Pointwise, $tI_{\{|X| > t\}} \leq |X|I_{\{|X| > t\}}$, and the expression on the right tends to zero and is dominated by the integrable $|X|$. Dominated convergence therefore gives

$$0 \leq tP(|X| > t) \leq \mathbb{E}[|X|I_{\{|X| > t\}}] \longrightarrow 0.$$

2. Let $X, Y \in L^2(\Omega, \mathcal{F}, P)$ with $\mathbb{E}X = \mathbb{E}Y = 0$ and $\mathbb{E}Y^2 > 0$. Find $a \in \mathbb{R}$ minimizing $\mathbb{E}(X - aY)^2$ and the minimum value. What inequality ensures that the minimum is nonnegative?

Answer. Expanding gives $\mathbb{E}X^2 - 2a\mathbb{E}(XY) + a^2\mathbb{E}Y^2$. Thus

$$a^* = \frac{\mathbb{E}(XY)}{\mathbb{E}Y^2}, \quad \min_a \mathbb{E}(X - aY)^2 = \mathbb{E}X^2 - \frac{[\mathbb{E}(XY)]^2}{\mathbb{E}Y^2} \geq 0.$$

The final inequality follows from Hölder's inequality with $p = q = 2$.

3. Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of random variables, and X be a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$. Suppose that $E(X_n) < \infty$ and that $X_n \rightarrow X$ uniformly in Ω , i.e., for all $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ (depending only on ϵ , not ω) such that for $n \geq N(\epsilon)$

$$|X_n(\omega) - X(\omega)| < \epsilon \text{ for all } \omega \in \Omega.$$

First, show that $E(X) < \infty$. Second, show that $\lim_{n \rightarrow \infty} E(X_n) = E\left(\lim_{n \rightarrow \infty} X_n\right) = E(X)$. Why is uniform convergence of $\{X_n\}_{n \in \mathbb{N}}$ needed? Wouldn't convergence on Ω be enough?

Answer: First, we need to show that $E(X) < \infty$. Now, $E(X) < \infty \iff \int_{\Omega} |X| dP < \infty$. Hence, choose any $\epsilon > 0$ such that $n \geq N(\epsilon)$ gives $|X_n(\omega) - X(\omega)| < \epsilon$ for all $\omega \in \Omega$. Then,

$$\begin{aligned} \int_{\Omega} |X| dP &= \int_{\Omega} |X - X_n + X_n| dP \leq \int_{\Omega} (|X - X_n| + |X_n|) dP \\ &\leq \int_{\Omega} (\epsilon + |X_n|) dP \text{ by uniform convergence} \\ &= \epsilon + \int_{\Omega} |X_n| dP < \infty \text{ by the fact that } P(\Omega) = 1 \text{ and } E(X_n) < \infty. \end{aligned}$$

Now, to show that $\lim_{n \rightarrow \infty} E(X_n) = E(X)$, consider

$$|E(X_n) - E(X)| = \left| \int_{\Omega} X_n dP - \int_{\Omega} X dP \right| = \left| \int_{\Omega} (X_n - X) dP \right| \leq \int_{\Omega} |X_n - X| dP \leq \epsilon$$

by uniform convergence.

Uniform convergence is needed because we need to bound $|X_n(\omega) - X(\omega)|$ for all ω *simultaneously*.