Homework 9 - Answers.

1.

1. $$E(\hat{E}(Y|x_1,x_2)|X) = E(\hat{\beta}_0|X) + E(\hat{\beta}_1|X)x_1 + E(\hat{\beta}_2|X)x_2$$
$$= \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

By the law of iterated expectations

$$E(\hat{E}(Y|x_1,x_2)) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 = E(Y|x_1,x_2)$$

It is unbiased.

2. $$\hat{E}(Y|x_1,x_2) = (1 \ \ x_1 \ \ x_2)\hat{\beta}, \quad \text{where } \hat{\beta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}$$
$$= x\hat{\beta} \quad \text{where } x = (1 \ \ x_1 \ \ x_2)$$

$$\hat{E}(Y|x_1,x_2) - E(Y|x_1,x_2) = x(\hat{\beta}-\beta)$$

$$V(\hat{E}(Y|x_1,x_2)|X) = E([x(\hat{\beta}-\beta)]^2|X)$$
$$= E(x(\hat{\beta}-\beta)(\hat{\beta}-\beta)^T x^T|X)$$
$$= x\,V(\hat{\beta}|X)\,x^T$$
$$= \sigma^2 x(X^TX)^{-1}x^T$$

3. $$\frac{x\hat{\beta} - x\beta}{\sqrt{\hat{\sigma}^2 x(X^TX)^{-1}x^T}} \sim t_{n-3}, \quad \text{where } \hat{\sigma}^2 = \frac{1}{n-3}\hat{U}^T\hat{U}$$

and $$\hat{U} = Y - X\hat{\beta}, \quad \text{with } Y^T = (Y_1 \cdots Y_n), \ X = \begin{bmatrix} 1 & X_{11} & X_{12} \\ \vdots & & \vdots \\ 1 & X_{n1} & X_{n2} \end{bmatrix}$$

For $\alpha \in (0,1)$

$$P\left(x\hat{\beta} + t^*_{n-3,\alpha/2}\sqrt{\hat{\sigma}^2 x(X^TX)^{-1}x^T} \leqslant x\beta \leqslant x\hat{\beta} + t^*_{n-3,1-\alpha/2}\sqrt{\hat{\sigma}^2 x(X^TX)^{-1}x^T}\right) = 1-\alpha$$

where $t^*_{d,p}$ is the p^{th} quantile associated a student-t distribution with d-degrees of freedom

2. Here is a list of the needed assumptions:

If we write the model in matrix form, i.e.,

$Y = X\beta + u$, then:

1. $E(u|X) = 0$. This guarantees that $E(R\hat{\beta}|X) = R\beta$ and $R\beta = r$ under Ho.

2. $E(uu^T|X) = \sigma^2 I_n$. This guarantees that $V(R\hat{\beta} - r|X) = \sigma^2(R(X^TX)^{-1}R^T)$

3. $U \sim N(0, \sigma^2 I)$, This guarantees that f the ratio of two (independent) χ^2 random variables divided by their degrees of freedom. m for the numerator and $n-(k+1)$ for the denominator.