Homework 6 - Answers.

1. Question 5

(i) $(AB)(AB)^{-1} = I \Rightarrow A^{-1}AB(AB)^{-1} = A^{-1} \Rightarrow B(AB)^{-1} = A^{-1} \Rightarrow$
$B^{-1}B(AB)^{-1} = B^{-1}A^{-1} \Rightarrow (AB)^{-1} = B^{-1}A^{-1}$

(ii) $(ABC)^{-1} = (AD)^{-1}$ if $D = BC$. By part (i), $(AD)^{-1} = D^{-1}A^{-1}$.
But again, by part (i), $D^{-1} = (BC)^{-1} = C^{-1}B^{-1}$! Hence,
$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Question 8.

Property 5. (p.756) says that for $A_{m\times n}$ and $b_{n\times 1}$, non random,
$V(Ay+b) = A\,V(y)\,A^T$.

By definition
$$\begin{aligned}
V(Ay+b) &= E\{[(Ay+b)-E(Ay+b)][Ay+b-E(Ay+b)]^T\} \\
&= E\{[Ay+b-AE(y)-b][Ay+b-AE(y)-b]^T\} \\
&= E\{A(y-E(y))[A(y-E(y))]^T\} \\
&= E\{A(y-E(y))(y-E(y))^T A^T\} \\
&= A\,E\{(y-E(y))(y-E(y))^T\}\,A^T \\
&= A\,V(y)\,A^T.
\end{aligned}$$

2. $AB = \begin{bmatrix} -35 & 16 \\ -56 & 28 \\ -73 & 38 \\ -48 & 36 \end{bmatrix}$. BA is not defined as the number of columns in B, i.e., 2 is different from the number of rows in A, i.e., 4

3. X is an $n \times K$ matrix.

$$X^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_{12} & X_{22} & \cdots & X_{n2} \\ \vdots & \vdots & & \vdots \\ X_{1K} & X_{2K} & \cdots & X_{nK} \end{bmatrix}$$

$$X^T X = \begin{bmatrix} n & \sum_{i=1}^{n} X_{i2} & \sum_{i=1}^{n} X_{i3} & \cdots & \sum_{i=1}^{n} X_{iK} \\ \sum_{i=1}^{n} X_{i2} & \sum_{i=1}^{n} X_{i2}^2 & \sum_{i=1}^{n} X_{i2}X_{i3} & \cdots & \sum_{i=1}^{n} X_{i2}X_{iK} \\ \vdots & \vdots & \ddots & & \vdots \\ \sum_{i=1}^{n} X_{iK} & \sum_{i=1}^{n} X_{iK}X_{i2} & \sum_{i=1}^{n} X_{iK}X_{i3} & \cdots & \sum_{i=1}^{n} X_{iK} \end{bmatrix}$$

4. (a) Let $Y^T = (Y_1 \cdots Y_n)_{1\times n}$, $X = \begin{bmatrix} 1 & X_{11} & X_{12} & X_{13} \\ 1 & X_{21} & X_{22} & X_{23} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & X_{n1} & X_{n2} & X_{n3} \end{bmatrix}_{n\times 4}$,

$\beta^T = (\beta_0 \ \beta_1 \ \beta_2 \ \beta_3)_{1\times 4}$ and $U^T = (U_1 \cdots U_n)$. Then,

$Y = X\beta + U$.

(b) $$S_n(\beta) = Y^TY - Y^TX\beta - \beta^TX^TY + \beta^TX^TX\beta$$
$$= Y^TY - 2Y^TX\beta + \beta^TX^TX\beta$$

$$\frac{\partial S_n(\beta)}{\partial \beta} = -2Y^TX + 2\beta^TX^TX$$

(c) $$\frac{\partial S_n(\hat\beta)}{\partial \beta} = 0 \Leftrightarrow -2Y^TX + 2\hat\beta^TX^TX = 0 \Leftrightarrow \hat\beta^TX^TX = Y^TX$$
$$\Leftrightarrow X^TX\hat\beta = X^TY$$

If X^TX is non-singular, i.e., if $(X^TX)^{-1}$ exists,

$$\hat\beta = (X^TX)^{-1}X^TY.$$

(d) Since, $E(U) = \begin{bmatrix} E(U_1) \\ \vdots \\ E(U_n) \end{bmatrix}$, $E(U) = 0$, given that $E(U_i) = 0$.

$$E(UU^T) = E\left\{\begin{bmatrix} U_1 \\ \vdots \\ U_n \end{bmatrix}[U_1 \cdots U_n]\right\} = E\left\{\begin{bmatrix} U_1^2 & U_1U_2 & \cdots & U_1U_n \\ U_2U_1 & U_2^2 & \cdots & U_2U_n \\ \vdots & \vdots & & \\ U_nU_1 & U_nU_2 & \cdots & U_n^2 \end{bmatrix}\right\} = \sigma^2 I_n$$

Since $E(U_i^2) = \sigma^2$ and $\{U_i\}_{i=1}^n$ is an independent sequence.