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Recovering distribution functions, interval probabilities and jumps via Fourier inversion theorems: new convergence bounds submitted to Kazakh Mathematical Journal

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Abstract. We provide new Fourier inversion theorems, with rates, that allow the recovery of a distribution function, associated interval probabilities, and jumps from the characteristic function. The results expand, improve, and clarify conditions imposed in our earlier work Mynbaev et al. (2022). First, we show that higher rates of convergence can be achieved by an appropriate choice of the regularization function, both for the recovery of interval probabilities and for jump discontinuities. Second, we propose a new inversion theorem for the recovery of the distribution function at points of continuity. Along the way, we clarify which of the conditions used previously are in fact necessary. The resulting theorems may be useful for constructing nonparametric estimators in errors-in-variables models where density functions may fail to exist.

Keywords: characteristic function, Fourier inversion theorem, distribution function, jumps of a distribution function.

1 Introduction

Fourier inversion theorems play an important role in probability theory (see, *e.g.* [6], [9] and [2]). They allow for the recovery of a distribution function, associated interval probabilities, and distribution jumps from the distribution's characteristic function (Fourier transform).

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Such recovery is useful in both theoretical and applied settings. For example, they are the main motivation for estimators that emerge in classical error-in-variable models (see [7], [3]).

Fourier inversion theorems are normally stated as limits of certain integrals, and until recently little was known about their rates of convergence. [1] provided the first attempt to obtain convergence rates for distribution functions and their jumps, and [8] obtained such rates for interval probabilities and jumps. The derivations in [8] are based on certain integral representations that involve a regularization function. The first contribution of this paper is to show that faster convergence rates can be obtained with a suitable choice of the regularization function, in both recovery of interval probabilities and jumps. The second contribution is an entirely new inversion theorem for recovering a distribution function. A byproduct of the contributions we make in this paper is to reveal which conditions imposed in [8] are necessary for their results.

The paper is organized as follows. Besides this introduction, section 2 provides a new theorem for the recovery of a distribution F at a point of continuity, gives rates of convergence, and describes a suitable selection of regularization function; section 3 studies the recovery of interval probabilities and, section 4 studies the recovery of jumps, both under suitably chosen regularization function. A brief conclusion section provides a summary of the results and directions for future work, including applications for our results.

We adopt the following notation throughout the paper: F denotes a distribution function; χ_S is the indicator function for the set S; $C(f)$ denotes the points of continuity of the function f; C denotes the space of uniformly bounded continuous functions on $\mathbb{R}$ with norm $\|f\|_C = \sup\limits_{t\in\mathbb{R}}|f(t)|$; $L_1(\mathbb{R})$ denotes the space of functions f on $\mathbb{R}$ with a finite norm

$$\|f\|_{L_1} = \int_{\mathbb{R}} |f(t)|dt;$$

$\mathcal{F}$ and $\mathcal{F}^{-1}$ denote, respectively, the Fourier and inverse Fourier transforms; for $f \in L_1(\mathbb{R})$,

$$\begin{aligned} (\mathcal{F}f)(t) &= \int_{\mathbb{R}} e^{ist} f(s)ds, \\ \left(\mathcal{F}^{-1}f\right)(t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ist} f(s)ds; \end{aligned}$$

for a distribution function F,

$$\begin{aligned} (\mathcal{F}F)(t) &= \int_{\mathbb{R}} e^{ist} dF(s), \\ \left(\mathcal{F}^{-1}F\right)(t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ist} dF(s). \end{aligned}$$

2 Recovering $F(x)$ for $x \in C(F)$

Perhaps the first attempt to recover a distribution function F from an inversion theorem was made by [5]. Using Lévy's inversion theorem, [5] obtained

$$F(x) = \frac{1}{2} + \frac{1}{2\pi} \lim_{\varepsilon \to 0,\ \lambda \to \infty} \int_{\varepsilon}^{\lambda} \frac{e^{itx}\phi(-t) - e^{-itx}\phi(t)}{it} dt \text{ for } \varepsilon,\ \lambda > 0 \text{ and } x \in C(F),$$

where $\phi := \mathcal{F}F$. As [10] pointed out, the integrand is not absolutely summable and the integral converges in the mean value sense. Furthermore, a rate of convergence was not given. More recently, [1] used Parseval's relation (see [4]) to obtain

$$F(x) = \lim_{t \to \infty} \frac{1}{c} \int_{(-\infty,x]} \int_{\mathbb{R}} e^{-ius} g(u/t)\phi(u)duds, \text{ for } t > 0 \text{ and } x \in C(F), \tag{1}$$

where $g \geq 0$ is any symmetric density (about 0), $\phi := \mathcal{F}F$ and $c := \int_{\mathbb{R}} \phi(s)ds$. When g is a standard normal density, [1, Theorem 1] gives lower and upper bounds on

$$\sup_{x \in \mathbb{R}} \left| \int_{(-\infty,x]} \frac{1}{c} \int_{\mathbb{R}} e^{-ius} g(u/t)\phi(u)duds - F(x) \right|,$$

providing a rate of convergence for the recovery of F via an inversion theorem.

In Theorem 4 below we give an inversion theorem in a different format, together with a rate of convergence. We start with a heuristic justification of our approach. Put

$$g_{\lambda,x}(t) := \chi_{(-1,0)}\left(\lambda(-t - x)\right), \text{ for } \lambda > 0,\ t, x \in \mathbb{R},$$

and noting that $g_{\lambda,x}(-t) = \chi_{\left(x - \frac{1}{\lambda}, x\right)}(t)$, we write

$$F(x) - F\left(x - 1/\lambda\right) = \int_{x-1/\lambda}^{x} dF(t) = \int_{\mathbb{R}} g_{\lambda,x}(-t)dF(t).$$

Defining

$$U_{\lambda,x}(s) := \left(g_{\lambda,x} * F\right)(s) = \int_{\mathbb{R}} g_{\lambda,x}(s - t)dF(t) \tag{2}$$

we have $F(x) - F(x - \frac{1}{\lambda}) = U_{\lambda,x}(0)$, and taking limits as $\lambda \to 0$ gives

$$F(x) = \lim_{\lambda \to 0} U_{\lambda,x}(0). \tag{3}$$

Since $g_{\lambda,x}$ is integrable, by Theorem 3.3.2 in [6] we have

$$(\mathcal{F}U_{\lambda,x})(t) = \left(\mathcal{F}g_{\lambda,x}\right)(t)\phi(t). \tag{4}$$

If the product on the right side of (4) is integrable, then from (3) and (4)

$$F(x) = \lim_{\lambda \to 0} (\mathcal{F}^{-1} \mathcal{F} U_{\lambda,x})(0) = \lim_{\lambda \to 0} \mathcal{F}^{-1} \left[(\mathcal{F} g_{\lambda,x}) \, \phi \right](0).$$

When the right side of (4) is not integrable, it is possible to regularize the integrand by multiplying it by $H(h\cdot)$ where $H \in L_1(\mathbb{R})$ and $h > 0$. H is chosen in such a way as to have

$$F(x) = \lim_{h,\lambda \to 0} \mathcal{F}^{-1} \left[H(h\cdot) \, (\mathcal{F} g_{\lambda,x}) \, \phi \right](0).$$

Lemma 1. *Let* $H \in L_1(\mathbb{R})$, $G([a,b]) = \frac{1}{2\pi} \int_a^b (\mathcal{F}H)(v) dv$ *for* $a < b$, $a, b \in \mathbb{R}$ *and define*

$$A_x(h,\lambda) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \frac{e^{it/\lambda} - 1}{it} H(ht) \phi(t) dt. \tag{5}$$

Then, for $h, \lambda > 0$ *and* $x \in \mathbb{R}$

$$A_x(h,\lambda) = \int_{\mathbb{R}} G\left(\left[\frac{t-x}{h}, \frac{t-x+1/\lambda}{h} \right] \right) dF(t).$$

Proof. Let $g_{\lambda,x}(t) = \chi_{(x-1/\lambda,x)}(-t)$ and note that

$$(\mathcal{F} g_{\lambda,x})(s) = \int_{\mathbb{R}} e^{ist} g_{\lambda,x}(t) \, dt = \int_{-x}^{-x+1/\lambda} e^{ist} \, dt = e^{-ixs} \frac{e^{is/\lambda} - 1}{is}.$$

The Fourier transform of $H(h\cdot)(\mathcal{F} g_{\lambda,x})\phi$ is given by

$$\mathcal{F}^{-1} \left[H(h\cdot)(\mathcal{F} g_{\lambda,x})\phi \right](s) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ist} H(ht) \left(e^{-ixt} \frac{e^{it/\lambda} - 1}{it} \right) \phi(t) \, dt,$$

and

$$\mathcal{F}^{-1} \left[H(h\cdot)(\mathcal{F} g_{\lambda,x})\phi \right](0) = \frac{1}{2\pi} \int_{\mathbb{R}} H(ht) e^{-ixt} \frac{e^{it/\lambda} - 1}{it} \phi(t) \, dt = A_x(h,\lambda).$$

Since $\phi(t) = \int_{\mathbb{R}} e^{itu} \, dF(u)$, we have

$$A_x(h,\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}} H(ht) e^{-ixt} \frac{e^{it/\lambda} - 1}{it} \left(\int_{\mathbb{R}} e^{itu} \, dF(u) \right) dt.$$

Assuming the integrability of $H(h\cdot)$, by Fubini's theorem we have

$$A_x(h,\lambda) = \int_{\mathbb{R}} \left(\frac{1}{2\pi} \int_{\mathbb{R}} H(ht) e^{-itx} \frac{e^{it/\lambda} - 1}{it} e^{itu} \, dt \right) dF(u).$$

To simplify the inner integral $\frac{1}{2\pi}\int_{\mathbb{R}} H(ht)e^{it(u-x)}\frac{e^{it/\lambda}-1}{it}\,dt$, let $v=ht$ and write

$$\frac{1}{2\pi}\int_{\mathbb{R}} H(v)e^{i(v/h)(u-x)}\frac{e^{i(v/h)/\lambda}-1}{i(v/h)}\frac{dv}{h}=\frac{1}{2\pi}\int_{\mathbb{R}} H(v)e^{iv(u-x)/h}\frac{e^{iv/(h\lambda)}-1}{iv}\,dv.$$

Since $\frac{e^{iv/(h\lambda)}-1}{iv}=\int_0^{1/(h\lambda)}e^{ivs}\,ds$, the inner integral is

$$\frac{1}{2\pi}\int_{\mathbb{R}} H(v)e^{iv(u-x)/h}\left(\int_0^{1/(h\lambda)}e^{ivs}\,ds\right)dv=\frac{1}{2\pi}\int_0^{1/(h\lambda)}\left(\int_{\mathbb{R}} H(v)e^{iv((u-x)/h+s)}\,dv\right)ds.$$

Since $\mathcal{F}H(v)=\int_{\mathbb{R}} H(u)e^{ivu}\,du$, we have

$$\int_{\mathbb{R}} H(v)e^{iv((u-x)/h+s)}\,dv=(\mathcal{F}H)\left(\frac{u-x}{h}+s\right).$$

Thus, if $w=\frac{u-x}{h}+s$

$$\begin{aligned}\frac{1}{2\pi}\int_0^{1/(h\lambda)}(\mathcal{F}H)\left(\frac{u-x}{h}+s\right)ds&=\frac{1}{2\pi}\int_{(u-x)/h}^{(u-x)/h+1/(h\lambda)}(\mathcal{F}H)(w)\,dw\\&=G\left(\left[\frac{u-x}{h},\frac{u-x+1/\lambda}{h}\right]\right).\end{aligned}$$

Therefore, $A_x(h,\lambda)=\int_{\mathbb{R}} G\left(\left[\frac{u-x}{h},\frac{u-x+1/\lambda}{h}\right]\right)dF(u)$, completing the proof. □

The next two lemmas show that if $x,\,y\in C(F)$, $\int_{\mathbb{R}} G\left(\left[\frac{t-y}{h},\frac{t-x}{h}\right]\right)dF(t)$ can approximate $F(x,y):=F(y)-F(x)$ given a suitable choice of G. To this end, let η be a function such that $\eta\ge 0$, $\operatorname{supp}\eta\subset[-1,1]$, $\frac{1}{2\pi}\int_{-1}^{1}\eta(t)dt=1$ and $\mathcal{F}^{-1}\eta\in L_1(\mathbb{R})$. Now, for $\delta\in(0,1)$ and $h>0$ define $p:=p_{\delta,h}$, $H:=H_{\delta,h}$ and $G:=G_{\delta,h}$ by

$$p(t)=\eta(h^{1-\delta}t)h^{1-\delta},\ H=\mathcal{F}^{-1}p,\ G([a,b])=\frac{1}{2\pi}\int_a^b(\mathcal{F}H)(t)dt=\frac{1}{2\pi}\int_a^b p(t)dt. \tag{6}$$

Then, $H\in L_1(\mathbb{R})$, $\|G\|_C=1$ and

$$p\ge 0,\ \operatorname{supp}p\subset\left[-h^{\delta-1},h^{\delta-1}\right],\ \frac{1}{2\pi}\int_{-h^{\delta-1}}^{h^{\delta-1}}p(t)dt=\frac{1}{2\pi}\int_{-1}^{1}\eta(t)dt=1. \tag{7}$$

Lemma 2. *Let $\delta\in(0,1)$ and $G([a,b])$ be as defined in (6). Then, for any F, $x,y\in C(F)$, $x<y$ we have*

$$\left|\int_{\mathbb{R}} G\left(\left[\frac{t-y}{h},\frac{t-x}{h}\right]\right)dF(t)-F(x,y)\right|\le 2\left[\omega(x,h^\delta)+\omega(y,h^\delta)\right],\ h>0,$$

where $\omega(x,h)=F(x+h)-F(x-h)$ for $x\in C(F)$.

Proof. We write

$$\int_{\mathbb{R}} G\left(\left[\frac{t-y}{h}, \frac{t-x}{h}\right]\right) dF(t) - F(x,y) = \int_{\mathbb{R}} G dF - \int_x^y dF$$
$$= \int_{-\infty}^{x-h^\delta} G dF + \int_{y+h^\delta}^{\infty} G dF + \int_{x+h^\delta}^{y-h^\delta} (G-1)\, dF + \int_{x-h^\delta}^{x+h^\delta} G dF$$
$$+ \int_{y-h^\delta}^{y+h^\delta} G dF - \int_x^{x+h^\delta} dF - \int_{y-h^\delta}^{y} dF$$

The integrals on the right-side of the last equality are denoted $I_1, \cdots, I_7$ in the order they are written.

1. In I_1, $t \le x - h^\delta$ and $\frac{t-x}{h} \le -h^{\delta-1}$ and, so $G := 0$ and $I_1 = 0$.
2. Similarly, $t \ge y + h^\delta$ and $\frac{t-y}{h} \ge h^{\delta-1}$ in I_2, so $G := 0$ and $I_2 = 0$.
3. In I_3 we have $x + h^\delta < t < y - h^\delta$, so $\frac{t-y}{h} < -h^{\delta-1} < h^{\delta-1} \le \frac{t-x}{h}$. This means that, for all t in this interval, the segment $\left[\frac{t-y}{h}, \frac{t-x}{h}\right]$ contains $\left[-h^{\delta-1}, h^{\delta-1}\right]$ and

$$G\left(\left[\frac{t-y}{h}, \frac{t-x}{h}\right]\right) = \frac{1}{2\pi}\int_{\frac{t-y}{h}}^{\frac{t-x}{h}} p(s)ds = \frac{1}{2\pi}\int_{-h^{\delta-1}}^{h^{\delta-1}} p(s)ds = 1,$$

so $I_3 = 0$.

Thus,

$$\int_{\mathbb{R}} G dF - F(x,y) = \int_{x-h^\delta}^{x+h^\delta} G dF + \int_{y-h^\delta}^{y+h^\delta} G dF - \int_x^{x+h^\delta} dF - \int_{y-h^\delta}^{y} dF. \tag{8}$$

This is bounded in absolute value by

$$(1 + \|G\|_C)\left[\omega(x, h^\delta) + \omega(y, h^\delta)\right] \le 2\left(\omega(x, h^\delta) + \omega(g, h^\delta)\right),$$

since $\|G\|_C = 1$. □

Using another version of (8), the bound obtained in Lemma 2 can be improved.

Lemma 3. *For any F and $\varepsilon \ge 1$, $G := G_{\varepsilon,h}$ and $x, y \in C(F)$ such that $x < y$*

$$\left|\int_{\mathbb{R}} G\left(\left[\frac{t-y}{h}, \frac{t-x}{h}\right]\right) dF(t) - F(x,y)\right| \le \omega(x, h^\varepsilon) + \omega(y, h^\varepsilon) \text{ for all } h \in (0,1].$$

Proof. We consider the first two integrals in (8), keeping the numbering of the integrals from the proof of Lemma 2. In

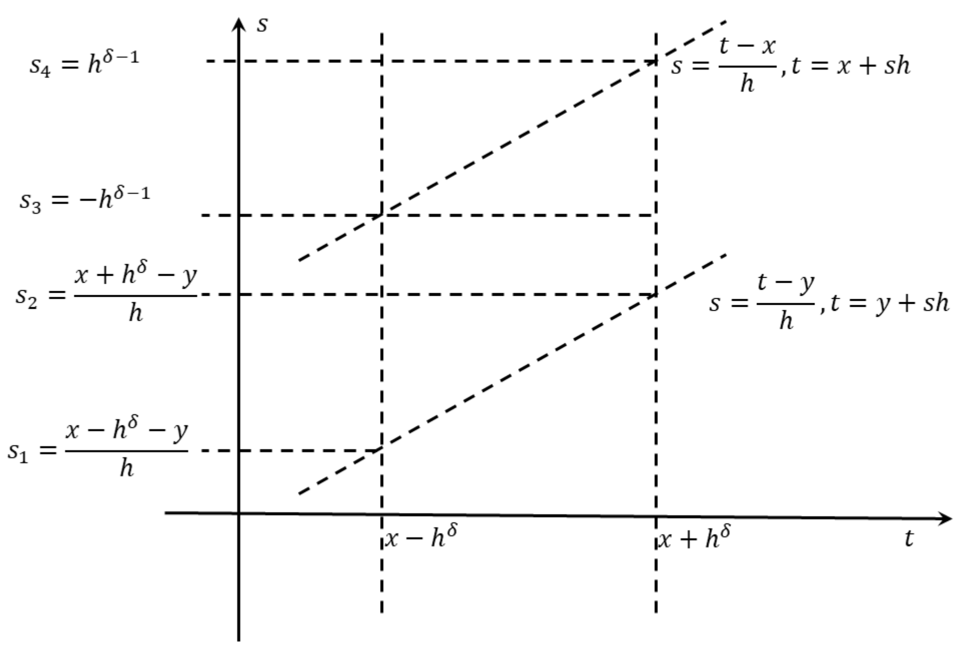
$$2\pi I_4 = \int_{x-h^\delta}^{x+h^\delta} \left(\int_{(t-y)/h}^{(t-x)/h} p(s)ds \right) dF(t)$$

we want to change the integration order, refer to Figure 1, where

$$s_1 = \frac{x-h^\delta-y}{h},\ s_2 = \frac{x+h^\delta-y}{h},\ s_3 = \frac{x-h^\delta-x}{h},\ s_4 = \frac{x+h^\delta-x}{h}.$$

Since $x<y$, we have $2h^\delta < y-x$, $s_3 - s_2 = \frac{y-x-2h^\delta}{h} > 0$. Hence, for small h it is true that

Fig. 1: Integration order for I_4



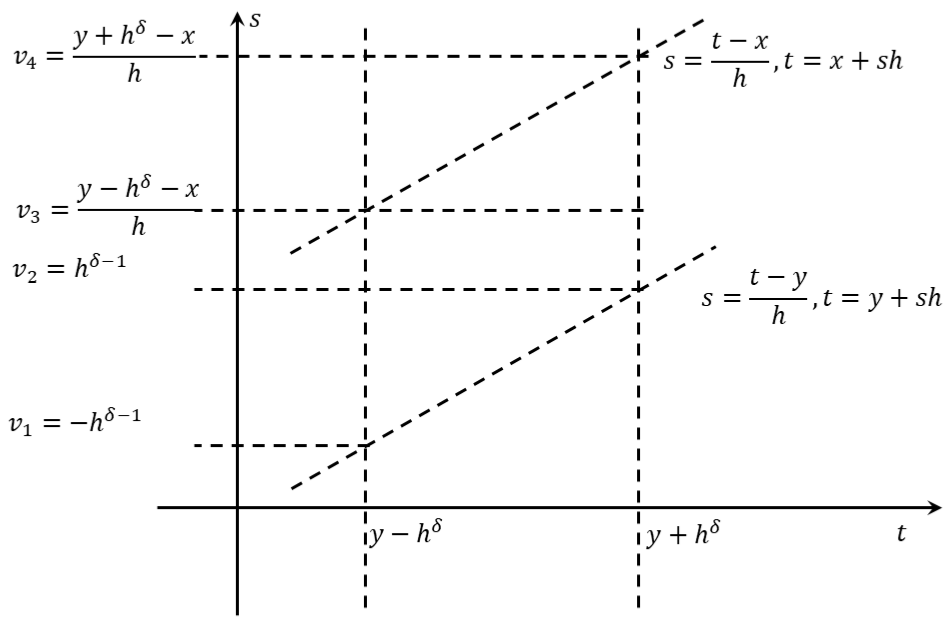
$s_1 < s_2 < s_3 = -h^{\delta-1} < s_4 = h^{\delta-1}$. Here $s_3 \to -\infty$ and $s_4 \to \infty$ as $h \to 0$. Therefore

$$\begin{aligned} 2\pi I_4 &= \int_{s_1}^{s_2} \left(\int_{x-h^\delta}^{y+sh} dF(t) \right) p(s)ds + \int_{s_2}^{s_3} \left(\int_{x-h^\delta}^{x+h^\delta} dF(t) \right) p(s)ds \\ &+ \int_{s_3}^{s_4} \left(\int_{x+sh}^{x+h^\delta} dF(t) \right) p(s)ds. \end{aligned}$$

Since p is zero outside $[s_3, s_4]$ we get

$$2\pi I_4 = \int_{s_3}^{s_4} \left(\int_{x+sh}^{x+h^\delta} dF(t) \right) p(s)ds = \int_{s_3}^{s_4} [F(x+h^\delta) - F(x+sh)]ds$$
$$= F(x+h^\delta) - \frac{1}{2\pi} \int_{-h^{\delta-1}}^{h^{\delta-1}} F(x+sh)p(s)ds. \quad (9)$$

Fig. 2: Integration order for I_5



For I_5, refer to Figure 2 where

$$v_1 = \frac{y - h^\delta - y}{h},\ v_2 = \frac{y + h^\delta - y}{h},\ v_3 = \frac{y - h^\delta - x}{h},\ v_4 = \frac{y + h^\delta - x}{h}.$$

For small h it is true that $v_1 = -h^{\delta-1} < v_2 = h^{\delta-1} < v_3 < v_4$. Therefore

$$\begin{aligned}
2\pi I_5 &= \int_{y-h^\delta}^{y+h^\delta} \left(\int_{(t-y)/h}^{(t-x)/h} p(s)ds \right) dF(t) \\
&= \int_{v_1}^{v_2} \left(\int_{y-h^\delta}^{y+sh} dF(t) \right) p(s)ds + \int_{v_2}^{v_3} \left(\int_{y-h^\delta}^{y+h^\delta} dF(t) \right) p(s)ds \\
&\quad + \int_{v_3}^{v_4} \left(\int_{x+sh}^{y+h^\delta} dF(t) \right) p(s)ds \\
&= \int_{v_1}^{v_2} [F(y+sh) - F(y-h^\delta)]ds. \qquad (10)
\end{aligned}$$

Combining (8)-(10) and given (7) we have

$$\begin{aligned}
\int_{\mathbb{R}} GdF - F(x,y) &= F(x+h^\delta) - \frac{1}{2\pi} \int_{-h^{\delta-1}}^{h^{\delta-1}} F(x+sh)p(s)ds \\
&\quad + \frac{1}{2\pi} \int_{-h^{\delta-1}}^{h^{\delta-1}} F(y+sh)p(s)ds \\
&\quad - F(y-h^\delta) - F(x+h^\delta) + F(x) - F(y) + F(y-h^\delta) \\
&= \frac{1}{2\pi} \int_{-h^{\delta-1}}^{h^{\delta-1}} [F(x) - F(x+sh)]p(s)ds \\
&\quad - \frac{1}{2\pi} \int_{-h^{\delta-1}}^{h^{\delta-1}} [F(y) - F(y+sh)]p(s)ds. \qquad (11)
\end{aligned}$$

Let p in (6) be given by $p_{\varepsilon,h}(t) = \eta(h^{1-\varepsilon}t)h^{1-\varepsilon}$ for $\varepsilon \geq 1$. Then $\operatorname{supp} p \subset \left[-h^{\varepsilon-1}, h^{\varepsilon-1}\right] \subset \left[-h^{\delta-1}, h^{\delta-1}\right]$ for $h \in (0,1]$. Since (7) is preserved, all calculations made up to this point remain valid. For the first integral on the right of Equation (11) we have

$$\left| \frac{1}{2\pi} \int_{-h^{\varepsilon-1}}^{h^{\varepsilon-1}} [F(x) - F(x+sh)]p(s)ds \right| \leq \omega(x, h^\varepsilon) \frac{1}{2\pi} \int_{-h^{\varepsilon-1}}^{h^{\varepsilon-1}} p(s)ds = \omega(x, h^\varepsilon).$$

This and a similar bound for the second term at the right of (11) complete the proof. □

Theorem 4. *a) Let G satisfy the condition*

$$\begin{gathered}
\lim_{a \to -\infty,\ b \to \infty} G([a,b]) = 1, \\
\lim_{b \to -\infty} G([a,b]) = 0, \ \lim_{a \to \infty} G([a,b]) = 0, \qquad \text{(G)} \\
G \textit{ is bounded in its domain}
\end{gathered}$$

and define

$$\phi_G(N) = \max\left\{\sup_{b<-N}|G([a,b])|,\ \sup_{a>N}|G([a,b])|,\ \sup_{a<-N,\ b>N}|G([a,b])-1|\right\}.$$

Then, for any F, $x \in C(F)$, $h \in (0,1]$, $\delta \in (0,1)$ and $\lambda > 0$ we have

$$|A_x(h,\lambda) - F(x)| \le \phi_G(h^{\delta-1}) + (1+\|G\|_C)\,\omega(x,h^\delta) + (2+\|G\|_C)\,F\left(x+1-\frac{1}{\lambda}\right). \tag{12}$$

b) Let $G([a,b]) = \frac{1}{2\pi}\int_a^b(\mathcal{F}H)(v)dv$ with $\mathcal{F}H \ge 0$. Then,

$$A_x(h,\lambda) \to F(x),\ \text{as } h,\lambda \to 0,\ \text{for any } F,\ x \in C(F) \tag{13}$$

is equivalent to a combination of $\mathcal{F}H \in L_1(\mathbb{R})$ and $\frac{1}{2\pi}\int_{\mathbb{R}}(\mathcal{F}H)(u)du = 1$, which, in turn, is equivalent to (G).
c) Let $\varepsilon \ge 1$. If $G = G_{\varepsilon,h}$ from Lemma 3 is used, then (12) can be improved as follows:

$$|A_x(h,\lambda) - F(x)| \le \omega(x,h^\varepsilon) + (2+\|G\|_C)F\left(x+1-\frac{1}{\lambda}\right),\ h \in (0,1),\ \lambda > 0.$$

Proof. a) Lemma 2 in [8] states that if G satisfies (G) and $\delta \in (0,1)$, then for all $h > 0$, F, $x, y \in C(F)$, such that $x < y$, one has

$$\left|\int_{\mathbb{R}} G\left(\left[\frac{t-y}{h},\frac{t-x}{h}\right]\right)dF(t) - F(x,y)\right| \le \phi_G(h^{\delta-1}) + (1+\|G\|_C)\left[\omega(x,h^\delta)+\omega(y,h^\delta)\right]. \tag{14}$$

Replacing the pair (x,y) with the pair $(x-1/\lambda, x)$ and using Lemma 1 we get

$$|A_x(h,\lambda) - F(x-1/\lambda,x)| \le \phi_G(h^{\delta-1}) + (1+\|G\|_C)\left[\omega(x-1/\lambda,h^\delta)+\omega(x,h^\delta)\right].$$

Since $0 < h \le 1$ we see that

$$\begin{aligned}
|A_x(h,\lambda) - F(x)| &\le \left|A_x(h,\lambda) - \left[F(x) - F\left(x-\frac{1}{\lambda}\right)\right]\right| + F\left(x-\frac{1}{\lambda}\right)\\
&\le \phi_G(h^{\delta-1}) + (1+\|G\|_C)\left[\omega(x,h^\delta) + F\left(x-\frac{1}{\lambda}+h^\delta\right)\right.\\
&\quad \left.-F\left(x-\frac{1}{\lambda}-h^\delta\right)\right] + F\left(x-\frac{1}{\lambda}\right)\\
&\le \phi_G(h^{\delta-1}) + (1+\|G\|_C)\,\omega(x,h^\delta) + (2+\|G\|_C)\,F\left(x-\frac{1}{\lambda}+1\right).
\end{aligned}$$

b) Suppose (13) is true. Then, by Lemma 1

$$\int_{\mathbb{R}} G\left(\left[\frac{t-x}{h}, \frac{t-x+1/\lambda}{h}\right]\right) dF(t) \to F(x),\ h, \lambda \to 0 \text{ for any } F,\ x \in C(F).$$

Taking F to be the Heaviside function and $x = 1$, we obtain

$$\int_{\mathbb{R}} G\left(\left[\frac{t-1}{h}, \frac{t-1+1/\lambda}{h}\right]\right) dF(t) = G\left(\left[-\frac{1}{h_n}, \frac{-1+1/\lambda_n}{h_n}\right]\right) \to 1$$

for any sequences $h_n \to 0$, $\lambda_n \to 0$. Here $-1/h_n \to -\infty$, $(-1+1/\lambda_n)/h_n \to \infty$. Denoting $\Delta_n = [-1/h_n, (-1+1/\lambda_n)/h_n]$, we get

$$\frac{1}{2\pi} \int_{\mathbb{R}} (\mathcal{F}H)(u)\chi_{\Delta_n}(u)du \to 1,$$

from the fact that $G([a,b]) = \frac{1}{2\pi}\int_a^b (\mathcal{F}H)(v)dv$. Since the sequence $(\mathcal{F}H)\chi_{\Delta_n}$ is non-decreasing, by the Beppo Levi's theorem this implies $\mathcal{F}H \in L_1(\mathbb{R})$ and $\frac{1}{2\pi}\int_{\mathbb{R}}(\mathcal{F}H)(u)du = 1$, which, in turn, implies (G). Conversely, if Equation (G) holds, then we have Equations (12) and (13). That, as we know, leads to $\mathcal{F}H \in L_1(\mathbb{R})$ and $\frac{1}{2\pi}\int_{\mathbb{R}}(\mathcal{F}H)(u)du = 1$.
c) Here, it suffices to use the bound from Lemma 3 instead of Equation (14). □

3 Recovering $F(x,y) := F(y) - F(x)$ for $x, y \in C(F)$

We start by defining

$$B_{x,y}(h) := \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-ixt} - e^{-iyt}}{it} H(ht)\phi(t)dt,\ H \in L_1(\mathbb{R}).$$

Theorem 5. *Suppose $G([a,b]) = \frac{1}{2\pi}\int_a^b (\mathcal{F}H)(v)dv$. a) If G satisfies (G), then for any F, $x, y \in C(F)$, $x < y$, and $\delta \in (0,1)$*

$$|B_{x,y}(h) - F(x,y)| \le \phi_G(h^{\delta-1}) + (1 + \|G\|_C)\left[\omega(x, h^\delta) + \omega(y, h^\delta)\right] \text{ for any } h > 0. \quad (15)$$

b) If $\mathcal{F}H \ge 0$, then the condition

$$B_{x,y}(h) \to F(x,y),\ h \to 0,\ \text{for any } F,\ x, y \in C(F),\ x < y, \quad (16)$$

is equivalent to the combination of $\mathcal{F}H \in L_1(\mathbb{R})$ and $\frac{1}{2\pi}\int_{\mathbb{R}}(\mathcal{F}H)(v)dv = 1$, which is equivalent to (G).
c) Let $\varepsilon \ge 1$. If G is as defined in Lemma 3, for all sufficiently small $h > 0$

$$|B_{x,y}(h) - F(x,y)| \le \omega(x, h^\varepsilon) + \omega(y, h^\varepsilon).$$

Proof. Part a) has been proved in [8, Theorem 7(i)].
b) Let $\mathcal{F}H \geq 0$ and suppose that (16) holds. Taking F to be the Heaviside function we have

$$B_{x,y}(h) = \int_{\mathbb{R}} G\left(\left[\frac{t-y}{h}, \frac{t-x}{h}\right]\right) dF(t) = G\left(\left[-\frac{y}{h}, -\frac{x}{h}\right]\right).$$

Setting $x = -1$, $y = 1$ gives

$$\frac{1}{2\pi}\int_{-1/h}^{1/h} (\mathcal{F}H)(v)dv = G\left(\left[-\frac{1}{h}, \frac{1}{h}\right]\right) \to F(x,y) = 1, \text{ as } h \to 0.$$

By Beppo Levi's Theorem $\mathcal{F}H \in L_1(\mathbb{R})$ and $\frac{1}{2\pi}\int_{\mathbb{R}}(\mathcal{F}H)(v)dv = 1$. This, in turn, implies (G). Conversely, (G) implies $\frac{1}{2\pi}\int_{\mathbb{R}}(\mathcal{F}H)(v)dv = 1$, while (16) follows from (15).

Finally, c) follows directly from Lemma 3. □

4 Recovering a jump p_x

The distribution F has a jump p_x at $x \in \mathbb{R}$ if $p_x := F(x) - \lim_{\varepsilon \downarrow 0} F(x - \varepsilon) > 0$. In this case we write $x \in J(F)$, the set of points where F has a jump. Let $K \in L_1(\mathbb{R})$ and $\int_{\mathbb{R}} K = 1$, then $W := \mathcal{F}K$ will be continuous and satisfy

$$W(0) = 1, \ \lim_{|x| \to \infty} W(x) = 0. \tag{17}$$

The continuity of W together with (17) implies the boundedness of W. In addition, letting

$$\omega_W(\varepsilon) := \sup_{|x| \leq \varepsilon} |W(x) - 1|, \quad \phi_W(N) := \sup_{|x| \geq N} |W(x)|, \ \delta_F(\varepsilon) := \int_{|t-x|<\varepsilon} dF(t) - p_x \geq 0,$$

we have that (17) implies $\omega_W(\varepsilon) \to 0$, as $\varepsilon \to 0$, and $\phi_W(N) \to 0$, as $N \to \infty$. Note also that $\lim_{\varepsilon \to 0} \delta_F(\varepsilon) = 0$ by continuity of probability measures.

For $h \in (0, 1)$ we define

$$C_x(h) := \int_{\mathbb{R}} e^{-itx}\phi(t)hK(ht)dt.$$

Theorem 6. *a) Let $\varepsilon_1 \in (0, 1)$, $\varepsilon_2 > \varepsilon_1$ and $K \in L_1(\mathbb{R})$. The condition $\int_{\mathbb{R}} K(t)dt = 1$ is equivalent to*

$$C_x(h) \to p_x \text{ for any } F, \ x \in J(F) \tag{18}$$

and also equivalent to

$$\begin{aligned} |C_x(h) - p_x| &\leq \omega_W(h^{\varepsilon_2 - \varepsilon_1})\left[p_x + \delta_F(h^{1-\varepsilon_1})\right] + (1 + \|W\|_C)\,\delta_F(h^{1-\varepsilon_1}) \\ &+ \phi_W(h^{-\varepsilon_1}) \text{ for all } h > 0, \ F, \ x \in J(F). \end{aligned} \tag{19}$$

b) Let ε, $h \in (0,1)$ and choose

$$W(t) := \begin{cases} 1, & |t| \leq (1-h)h^{-\varepsilon}; \\ 0, & |t| \geq h^{-\varepsilon}; \\ h^{\varepsilon-1}(h^{-\varepsilon} - |t|), & \textit{otherwise.} \end{cases} \tag{20}$$

such that $K := \mathcal{F}^{-1}W$ is integrable. Then, we have

$$\delta_F((1-h)h^{1-\varepsilon}) \leq C_x(h) - p_x \leq \delta_F(h^{1-\varepsilon}) \textit{ for all } h > 0,\ F,\ x \in J(F).$$

Proof. a) If $\int_{\mathbb{R}} K(t)dt = 1$ then $W = \mathcal{F}K$ is continuous and satisfies (17). From [8, Equation (A12)]

$$C_x(h) = \int_{\mathbb{R}} W\left(\frac{t-x}{h}\right) dF(t), \tag{21}$$

and by [8, Lemma 3 (i)] we obtain (19). Furthermore, (19) implies (18). Conversely, if (18) is true, then using (21) we can prove that $W(0) = 1$ by choosing F to be the Heaviside function. Thus, (17) holds and implies (19).

b) For the upper bound, we write

$$C_x(h) - p_x = \int_{\mathbb{R}} W\left(\frac{t-x}{h}\right) dF(t) - p_x = \int_{|t-x|<h^{1-\varepsilon}} W\left(\frac{t-x}{h}\right) dF(t) - p_x$$
$$+ \int_{|t-x|\geq h^{1-\varepsilon}} W\left(\frac{t-x}{h}\right) dF(t).$$

Here, $I_2 := \int_{|t-x|\geq h^{1-\varepsilon}} W\left(\frac{t-x}{h}\right) dF(t) = 0$ by (20). Since $W \leq 1$ everywhere,

$$\begin{aligned} C_x(h) - p_x &= \int_{|t-x|<h^{1-\varepsilon}} W\left(\frac{t-x}{h}\right) dF(t) - p_x = \\ &\leq \int_{|t-x|<h^{1-\varepsilon}} dF(t) - p_x = \delta_F(x, h^{1-\varepsilon}). \end{aligned}$$

For the lower bound, we continue to have $I_2 := 0$. Reducing the domain of integration we get by (20)

$$C_x(h) - p_x \geq \int_{|t-x|<(1-h)h^{1-\varepsilon}} W\left(\frac{t-x}{h}\right) dF(t) - p_x = \delta_F(x, (1-h)h^{1-\varepsilon}).$$

□

5 Conclusion

We provide new Fourier inversion theorems, with rates, that allow the recovery of a distribution function, associated interval probabilities, and jumps from the characteristic function. The results expand, improve and clarify conditions imposed in our earlier work [8]. The results may prove useful in motivating deconvolution estimators for distribution functions, associated interval probabilities and jumps in classical error-in-variable models when densities may not exist.

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Мыңбаев Қ. Т., Мартинс-Филью К. Б. ФУРЬЕ ИНВЕРСИЯ ТЕОРЕМАЛАРЫ АРҚЫЛЫ ТАРАЛУ ФУНКЦИЯЛАРЫ МЕН ОЛАРДЫҢ СЕКІРІСТЕРІН ЖӘНЕ ИНТЕРВАЛДЫҚ ЫҚТИМАЛДЫҚТАРДЫ ҚАЙТА ҚҰРУ: ЖИНАҚТЫЛЫҚ ЖЫЛДАМДЫҒЫНЫҢ ЖАҢА БАҒАЛАРЫ

Біз Фурье түрлендіруінің жаңа инверсия теоремаларын береміз. Таралу функциясы мен оның үзілістері, аралық ықтималдықтар қарастырылады және конвергенция жылдамдығы бағаланады. Нәтижелер біздің бұрынғы жұмысымызда қойылған шарттарды кеңейтеді, жақсартады және нақтылайды. Біріншіден, регуляризациялаушы функцияны дұрыс таңдау арқылы интервалдық ықтималдықтарды да, секірістерді де қалпына келтіру кезінде жинақталудың неғұрлым жоғары жылдамдықтарына қол жеткізуге болатынын көрсетеміз. Екіншіден, үлестірім функциясын үздіксіздік нүктелерінде қалпына келтіруге арналған мүлде жаңа инверсия теоремасын ұсынамыз. Сонымен қатар, бұрын қолданылған шарттардың қайсысы шынымен қажетті екендігін нақтылаймыз. Алынған теоремалар айнымалыларда қателіктері бар моделдерде, үлестірім тығыздықтары мүлде болмауы мүмкін жағдайларда, параметрлік емес бағалауларды құру үшін пайдалы болуы мүмкін.

Түйін сөздер: сипаттамалық функция, Фурье инверсия теоремасы, таралу функциясы, таралу функциясының үзілістері.

Мынбаев К. Т., Мартинс-Филью К. Б. ВОССТАНОВЛЕНИЕ ФУНКЦИЙ РАСПРЕДЕЛЕНИЯ И ИХ СКАЧКОВ, А ТАКЖЕ ИНТЕРВАЛЬНЫХ ВЕРОЯТНОСТЕЙ С ПОМОЩЬЮ ТЕОРЕМ ОБРАЩЕНИЯ ФУРЬЕ: НОВЫЕ ОЦЕНКИ СКОРОСТИ СХОДИМОСТИ

Мы доказываем новые теоремы об обращении преобразования Фурье, с оценками скорости сходимости, которые позволяют восстановить функцию распределения, соответствующие интервальные вероятности и скачки характеристической функции. Результаты расширяют и уточняют условия, полученные в нашей предыдущей работе. Во-первых, мы показываем, что более высокие скорости сходимости могут быть достигнуты при подходящем выборе регуляризующей функции как для восстановления интервальных вероятностей, так и для скачков. Во-вторых, мы предлагаем новую теорему обращения для восстановления функции распределения в точках непрерывности. Попутно мы проясняем, какие из условий, использованных ранее, являются необходимыми. Полученные теоремы могут быть полезны для построения непараметрических оценок в моделях с ошибками в переменных, когда плотности распределения могут не существовать.

Ключевые слова: характеристическая функция, теорема обращения Фурье, функция распределения, скачки функции распределения.