Homework 1.

1. $$E(X) = \int_0^1 x\, f(x)\, dx$$

$$= \int_0^1 x\, 2(1-x)\, dx = 2\int_0^1 x(1-x)\, dx = 2\int_0^1 (x - x^2)\, dx$$

$$= 2\left\{\int_0^1 x\, dx - \int_0^1 x^2 dx\right\} = 2\left\{ \Big|_0^1 \frac{x^2}{2} - \Big|_0^1 \frac{x^3}{3} \right\}$$

$$= 2\left\{ \frac{1}{2} - \frac{1}{3} \right\} = 2\,\frac{1}{6} = \frac{1}{3}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_0^1 x^2\, 2(1-x)\, dx = 2\int_0^1 (x^2 - x^3)\, dx$$

$$= 2\left\{ \Big|_0^1 \frac{x^3}{3} - \Big|_0^1 \frac{x^4}{4} \right\}.$$

$$= 2\left\{ \frac{1}{3} - \frac{1}{4} \right\} = 2 \cdot \frac{1}{12} = \frac{1}{6}$$

So, $V(X) = \frac{1}{6} - \left[\frac{1}{3}\right]^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$

2. (1.) Since different questions do not measure a student's ability perfectly, the same student can get different SAT scores on different dates of the exam.

(2.) If $X \sim N(5,4)$ then

- $X \leq 6 \Leftrightarrow X-5 \leq 1 \Leftrightarrow \frac{X-5}{2} \leq \frac{1}{2}$.

But $Z = \frac{X-5}{2}$ is a random variable with standard normal density. Hence,

$$P(X \leq 6) = P(Z \leq \tfrac{1}{2}) = 0.6915$$

- $X > 4 \Leftrightarrow X-5 > -1 \Leftrightarrow \frac{X-5}{2} > -\frac{1}{2}$

$$P(X>4) = P(Z > -\tfrac{1}{2}) = 1 - P(Z \leq -\tfrac{1}{2})$$
$$= 0.6915$$

- $|X-5| > 1 \Leftrightarrow X-5>1$ or $X-5 < -1$

$\Leftrightarrow X > 6$ or $X < 4$

$$P(|X-5|>1) = P(X>6) + P(X<4).$$
$$= 1 - P(X \leq 6) + 1 - P(X \geq 4)$$
$$= 2 - 0.6915 - 0.6915$$
$$= 0.617$$

(4) $X \in [0,1]$

$$P(X \leq x) = F(x) = 3x^2 - 2x^3 \quad \text{for} \quad 0 \leq x \leq 1$$

$$P(X \geq 0.6) = 1 - P(X \leq 0.6)$$

$$= 1 - \left[3(0.6)^2 - 2(0.6)^3\right]$$

$$= 0.352$$

(6) $$E(X) = \int_0^3 x \, \frac{1}{9} x^2 \, dx = \frac{1}{9} \Big|_0^3 \frac{x^4}{4} = \frac{9}{4}$$

11 (i) $E(X) = \frac{1}{2}1 + \frac{1}{2}(-1) = 0$

$E(X^2) = \frac{1}{2}1^2 + \frac{1}{2}(-1)^2 = 1$

(ii) $E(X) = \frac{1}{2}1 + \frac{1}{2}2 = 1.5$

$E\left(\frac{1}{X}\right) = \frac{1}{2}1 + \frac{1}{2}\frac{1}{2} = \frac{3}{4}$

(iii) In (i), $g(x) = x^2$ and $E(g(X)) = 1 \neq g(E(X)) = 0$

In (ii), $g(x) = \frac{1}{x}$ and $E(g(X)) = \frac{3}{4} \neq \frac{1}{E(X)} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$