Homework 2

1. If X and Y are independent, the joint density of X and Y, $f_{XY}(x,y) = f_X(x) f_Y(y)$.

Since $f_{Y|X=x}(y) = \dfrac{f_{XY}(x,y)}{f_X(x)}$, then $f_{Y|X=x}(y) = f_Y(y)$.

Hence, $E(Y|X) = \int y \, f_{Y|X=x}(y)\, dy = \int y \, f_Y(y)\, dy = E(Y)$

2. $Cov(X,Y) = E(XY) - E(X)E(Y)$. But by the Law of Iterated expectations

$$\begin{aligned} E(XY) &= E(E(XY|X)) \\ &= E(X\,E(Y|X)) \\ &= E(X\,E(Y)) \text{ by assumption} \\ &= E(Y)E(X). \end{aligned}$$

Hence, $Cov(X,Y) = 0$.

3. Problem 10 in the textbook.

(i) $E(GPA \mid SAT = 800) = 0.7 + 0.002 \times 800 = 2.3$

$E(GPA \mid SAT = 1400) = 0.7 + 0.002 \times 1400 = 3.5$

(ii) $E(E(GPA \mid SAT)) = E(GPA)$ and

$$\begin{aligned} E(GPA) &= 0.7 + 0.002\, E(SAT) \\ &= 0.7 + 0.002 \times 1{,}100 = 2.9 \end{aligned}$$

(iii) No. Only the expected value will be equal to the number in part (ii)

4. Problem 1 in Appendix C in the textbook.

Each Y_i for $i = 1, 2, 3, 4$ is such that $E(Y_i) = \mu$, $V(Y_i) = \sigma^2$

(i) Let $\bar{Y} = \frac{1}{4} \Sigma_{i=1}^4 Y_i$. Then, $E(\bar{Y}) = \frac{1}{4} \sum_{i=1}^{4} E(Y_i) = \mu$.

$$V(\bar{Y}) = V\left(\frac{1}{4} \Sigma_{i=1}^4 Y_i\right) = \frac{1}{16} V(\Sigma_{i=1}^4 Y_i)$$

$$= \frac{1}{16} \Sigma_{i=1}^4 V(Y_i) \quad \text{by independence}$$

$$= \frac{1}{16} \cdot 4\sigma^2 = \sigma^2/4.$$

(ii) $E(W) = \frac{1}{8}\mu + \frac{1}{8}\mu + \frac{1}{4}\mu + \frac{1}{2}\mu = \mu$

Since $E(W) = \mu$, W is unbiased.

$$V(W) = \frac{1}{8^2}\sigma^2 + \frac{1}{8^2}\sigma^2 + \frac{1}{4^2}\sigma^2 + \frac{1}{2^2}\sigma^2$$

$$= 0.3438\,\sigma^2$$

Since $V(W) > V(\bar{Y}) = 0.25\,\sigma^2$. $\bar{Y}$ is preferred.