

Question 4: (Appendix C)

$$(i) \quad Z = \frac{Y}{X} \Rightarrow E(Z|X) = \frac{1}{X} E(Y|X) = \frac{\theta X}{X} = \theta$$

$$E(Z) = E(E(Z|X)) = E(\theta) = \theta$$

$$(ii) \quad E(W_1 | X_1, \dots, X_n) = \frac{1}{n} \sum_{i=1}^n E(Y_i | X_i) = \frac{1}{n} \sum_{i=1}^n \frac{1}{X_i} \theta X_i \\ = \theta$$

$$(iii) \quad W_2 = \frac{\frac{1}{n} \sum_{i=1}^n Y_i}{\frac{1}{n} \sum_{i=1}^n X_i} = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i} \neq \sum_{i=1}^n \frac{Y_i}{X_i}$$

$$E(W_2 | X_1, \dots, X_n) = \frac{1}{\frac{1}{n} \sum_{i=1}^n X_i} \frac{1}{n} \sum_{i=1}^n E(Y_i | X_i) \\ = \frac{1}{\frac{1}{n} \sum_{i=1}^n X_i} \frac{1}{n} \sum_{i=1}^n \theta X_i = \theta.$$

Question 1 (Appendix D)

$$(i) \quad AB = \begin{pmatrix} 2 & -1 & 7 \\ -4 & 5 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 6 \\ 1 & 8 & 0 \\ 3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 20 & -6 & 12 \\ 5 & 36 & -24 \end{pmatrix}$$

(ii) BA is not defined.

Question 2 (Appendix D)

$$A = \begin{bmatrix} A_{11} & 0 & \dots & 0 \\ 0 & A_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & A_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & 0 & \dots & 0 \\ 0 & B_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{nn} \end{bmatrix}.$$

$$AB = \begin{pmatrix} A_{11}B_{11} & 0 & \dots & 0 \\ 0 & A_{22}B_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & A_{nn}B_{nn} \end{pmatrix} \quad \text{and}$$

$$BA = \begin{pmatrix} A_{11}B_{11} & 0 & \dots & 0 \\ 0 & A_{22}B_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & A_{nn}B_{nn} \end{pmatrix}$$

Question 3:

A is symmetric if $A_{ij} = A_{ji}$ for all $i = 1, \dots, K$ and $j = 1, 2, \dots, K$. Let $X_{n \times K} = [X_{ij}]_{i=1, j=1}^{n, K}$

$$X^T X = \begin{bmatrix} \sum_{i=1}^n X_{i1}^2 & \sum_{i=1}^n X_{i1}X_{i2} & \dots & \sum_{i=1}^n X_{i1}X_{iK} \\ \sum_{i=1}^n X_{i2}X_{i1} & \sum_{i=1}^n X_{i2}^2 & \dots & \sum_{i=1}^n X_{i2}X_{iK} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^n X_{iK}X_{i1} & \sum_{i=1}^n X_{iK}X_{i2} & \dots & \sum_{i=1}^n X_{iK}^2 \end{bmatrix}_{K \times K}$$

But the (k, l) position in this matrix is equal to the (l, k) position. i.e., $\sum_{i=1}^n X_{ik}X_{il} = \sum_{i=1}^n X_{il}X_{ik}$.

Question 4 (Appendix D)

(i) The trace of a square matrix is the sum of the elements in the main diagonal. If A is of dimension $n \times m$, $A^T A$ is $m \times m$, and from

Question 3,

$$\text{tr}(A^T A) = \sum_{j=1}^m \sum_{i=1}^n x_{ij}^2 \quad \checkmark$$

$A A^T$ is $n \times n$ and

$$\text{tr}(A A^T) = \sum_{i=1}^n \sum_{j=1}^m x_{ij}^2 \quad \checkmark$$

$$(ii) \text{tr}(A^T A) = 14 = \text{tr}(A A^T)$$

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nm} \end{bmatrix}_{n \times m}$$

$$A^T = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1m} & A_{2m} & \dots & A_{nm} \end{bmatrix}_{m \times n}$$

$$A^T A = \begin{bmatrix} \sum_{j=1}^n A_{j1}^2 & \sum_{j=1}^n A_{j1} A_{j2} & \dots & \sum_{j=1}^n A_{j1} A_{jm} \\ \sum_{j=1}^n A_{j2} A_{j1} & \sum_{j=1}^n A_{j2}^2 & \dots & \sum_{j=1}^n A_{j2} A_{jm} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^n A_{jm} A_{j1} & \sum_{j=1}^n A_{jm} A_{j2} & \dots & \sum_{j=1}^n A_{jm}^2 \end{bmatrix}_{m \times m}$$

$$\text{tr}(A^T A) = \sum_{i=1}^m \sum_{j=1}^n A_{ji}^2$$

$$A A^T$$

2.(a) let $S_n(\beta) = \sum_{i=1}^n (Y_i - \beta X_i)^2$.

$$\frac{dS_n(\beta)}{d\beta} = \sum_{i=1}^n 2(Y_i - \beta X_i)(-X_i) = -2 \sum_{i=1}^n (Y_i - \beta X_i) X_i$$

Setting $\frac{dS_n}{d\beta}(\hat{\beta}) = 0$, we have $\sum_{i=1}^n (Y_i X_i - \hat{\beta} X_i^2) = 0$.

The last equality implies $\hat{\beta} = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i^2}$.

$$E(\hat{\beta} | X_1, \dots, X_n) = \frac{1}{\sum_{i=1}^n X_i^2} \sum_{i=1}^n X_i E(Y_i | X_1, \dots, X_n)$$

$$= \frac{1}{\sum_{i=1}^n X_i^2} \sum_{i=1}^n X_i^2 \beta = \beta.$$

Hence, by the Law of Iterated expectations

$$E(\hat{\beta}) = E(E(\hat{\beta} | X_1, \dots, X_n)) = E(\beta) = \beta.$$

Now,

$$\hat{\beta} = \frac{1}{\sum_{i=1}^n X_i^2} \sum_{i=1}^n X_i (X_i \beta + U_i) = \beta + \frac{\sum_{i=1}^n X_i U_i}{\sum_{i=1}^n X_i^2}, \quad \text{i.e.,}$$

$$\hat{\beta} - \beta = \frac{\sum_{i=1}^n X_i U_i}{\sum_{i=1}^n X_i^2}$$

$$V(\hat{\beta} - \beta | X_1, \dots, X_n) = \frac{1}{\left(\sum_{i=1}^n X_i^2\right)^2} \sum_{i=1}^n X_i^2 V(U_i | X_1, \dots, X_n)$$

by the fact that $\{U_i\}_{i=1}^n$ is independent

$$= \sigma^2 / \sum_{i=1}^n X_i^2$$

Since as $n \rightarrow \infty$, $\sum_{i=1}^n X_i^2 \rightarrow \infty$

$$V(\hat{\beta} - \beta | X_1, \dots, X_n) = V(\hat{\beta} | X_1, \dots, X_n) \rightarrow 0$$

This is a desirable property. As the sample size grows the distribution of $\hat{\beta}$ conditional on $X_1, \dots, X_n$ gets closer to a degenerate distribution centered at β .

(b) Let $\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \hat{\beta} X_i)^2$ and consider

$$\begin{aligned} \sum_{i=1}^n (Y_i - \hat{\beta} X_i)^2 &= \sum_{i=1}^n (\beta X_i + U_i - \hat{\beta} X_i)^2 \\ &= \sum_{i=1}^n (-(\hat{\beta} - \beta) X_i + U_i)^2 \\ &= \sum_{i=1}^n ((\hat{\beta} - \beta)^2 X_i^2 + U_i^2 - 2(\hat{\beta} - \beta) X_i U_i) \\ &= (\hat{\beta} - \beta)^2 \sum_{i=1}^n X_i^2 + \sum_{i=1}^n U_i^2 - 2(\hat{\beta} - \beta) \sum_{i=1}^n X_i U_i \end{aligned}$$

But since $\sum_{i=1}^n X_i U_i = (\hat{\beta} - \beta) \sum_{i=1}^n X_i^2$,

$$\sum_{i=1}^n (Y_i - \hat{\beta} X_i)^2 = -(\hat{\beta} - \beta)^2 \sum_{i=1}^n X_i^2 + \sum_{i=1}^n U_i^2.$$

Then, taking expectations conditional on $X_1, \dots, X_n$

$$\begin{aligned} E\left(\sum_{i=1}^n (Y_i - \hat{\beta} X_i)^2 | X_1, \dots, X_n\right) &= -\frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n X_i^2} \sigma^2 + n \sigma^2 \\ &= \sigma^2(n-1) \end{aligned}$$

Hence, $E(\hat{\sigma}^2 | X_1, \dots, X_n) = \frac{1}{n-1} \sigma^2(n-1) = \sigma^2$.

and $E(\hat{\sigma}^2) = E(E(\hat{\sigma}^2 | X_1, \dots, X_n)) = \sigma^2$

c) Choose $\alpha = 0.05$

$$P\left(t_{n-1, \frac{\alpha}{2}} \leq \frac{\hat{\beta} - \beta}{\hat{\sigma}_{\hat{\beta}}} \leq t_{n-1, 1 - \frac{\alpha}{2}}\right) = 1 - \alpha$$

where,
$$\hat{\sigma}_{\hat{\beta}}^2 = \frac{\hat{\sigma}^2}{\sum_{i=1}^n (x_i - \bar{x})^2}.$$

• Under $H_0: \beta = 0$, $\hat{t} = \frac{\hat{\beta}}{\hat{\sigma}_{\hat{\beta}}} \sim t_{n-1}$. For a two-sided

test find the critical value t^* such that

$$P(|t| \leq t^*) = P(-t^* \leq t \leq t^*) = 1 - \alpha$$

If $|\hat{t}| < t^*$ accept (fail to reject) $H_0: \beta = 0$.

If $|\hat{t}| > t^*$ reject H_0 .

• Under $H_1: \beta = 1$, $\tilde{t} = \frac{\hat{\beta} - 1}{\hat{\sigma}_{\hat{\beta}}} \sim t_{n-1}$, use

the same decision rule as given in the previous test, substituting $\hat{t}$ with $\tilde{t}$.