The main steps in proving Carathéodory's Extension Theorem

Define the set function $\mu^* : 2^{\mathbb{X}} \to [0, \infty]$ such that for any $A \in 2^{\mathbb{X}}$

$$\mu^*(A) = \inf_{C(A)} \left\{ \sum_{j \in \mathbb{N}} \mu(C_j) \right\},$$

with $C(A) = \left\{ \{C_j\}_{j\in\mathbb{N}} \subset \mathcal{S} : A \subset \bigcup_{j\in\mathbb{N}} C_j \right\}$. $\bigcup_{j\in\mathbb{N}} C_j$ is a countable cover for A and μ^* is well defined because μ is defined on $\mathcal{S}$. If $C(A) = \emptyset$, then $\mu^*(A) = \infty$.

Step 1: Prove that μ^* is an *outer measure*, i.e., a function that satisfies

1. $\mu^*(\emptyset) = 0$,
2. $A \subset B \implies \mu^*(A) \le \mu^*(B)$,
3. $\mu^*\left(\bigcup_{j\in\mathbb{N}} A_j\right) \le \sum_{j\in\mathbb{N}} \mu^*(A_j)$.

Step 2: Prove that $\mu^*(S) = \mu(S)$ for all $S \in \mathcal{S}$. That is, μ^* is an extension of μ to $2^{\mathbb{X}}$.

Step 3: Show that $\mathcal{S} \subset \mathcal{A}^*$ where

$$\mathcal{A}^* = \{A \subset \mathbb{X} : \mu^*(A \cap Q) + \mu^*(A^c \cap Q) = \mu^*(Q), \forall\, Q \subset \mathbb{X}\}.$$

The sets in $\mathcal{A}^*$ are called the μ^*-measurable sets. The proof reveals that we cannot expect μ^* to be σ-additive outside $\mathcal{A}^*$.

Step 4: Show that $\mathcal{A}^*$ is a σ-algebra and μ^* is a measure on $(\mathbb{X}, \mathcal{A}^*)$. Since $\sigma(\mathcal{S}) \subset \mathcal{A}^*$, μ^* is a measure on $(\mathbb{X}, \sigma(\mathcal{S}))$ which extends μ from $\mathcal{S}$ to $\sigma(\mathcal{S})$.