Chapter 3

Measurable functions

In this chapter we will define measurable functions and study some of their properties. We start with the following definition.

Definition 3.1. *Let* $(\mathbb{X}, \mathcal{F})$ *and* $(\mathbb{E}, \mathcal{E})$ *be two measurable spaces. A function* $f : (\mathbb{X}, \mathcal{F}) \to (\mathbb{E}, \mathcal{E})$ *is said to be* $\mathcal{F} - \mathcal{E}$ *measurable if for all* $A \in \mathcal{E}$, $f^{-1}(A) \in \mathcal{F}$.

Remark 3.1.

1. *Since* $f^{-1}(\mathcal{E})$ *is a* σ*-algebra, measurability of* f *is equivalent to stating that* $f^{-1}(\mathcal{E}) \subset \mathcal{F}$. *It is standard notation to write* $\sigma(f) := f^{-1}(\mathcal{E})$ *and call this* σ*-algebra the* σ*-algebra generated by* f.

2. *If* $\mathbb{X} := \Omega$, $(\Omega, \mathcal{F}, P)$ *is a probability space and* f *is* $\mathcal{F} - \mathcal{E}$ *measurable, we say that* f *is a random element. If, in addition,* $(\mathbb{E}, \mathcal{E}) := (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *we will refer to* $f : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *as a random variable. We will normally represent random elements or random variables by uppercase roman letters, e.g.,* X *or* Y.

The next theorem shows that measurability of a function f can be established by examining inverse images of sets in a collection that generates the measurable sets associated with the co-domain of f.

Theorem 3.1. *Let* $\mathcal{C}$ *be a collection of subsets of* $\mathbb{E}$ *such that* $\sigma(\mathcal{C}) = \mathcal{E}$. *Then,* $f : (\mathbb{X}, \mathcal{F}) \to (\mathbb{E}, \mathcal{E})$ *is* $\mathcal{F} - \mathcal{E}$ *measurable* $\iff f^{-1}(\mathcal{C}) \subset \mathcal{F}$.

Proof. ($\Longrightarrow$) Assume f is $\mathcal{F}-\mathcal{E}$ measurable. f measurable $\iff$ for all $A \in \mathcal{E}$, $f^{-1}(A) \in \mathcal{F}$. In particular, let A be an element of $\mathcal{C}$, then $f^{-1}(A) \in \mathcal{F}$, hence $f^{-1}(\mathcal{C}) \subset \mathcal{F}$.

($\Longleftarrow$) Assume that $f^{-1}(\mathcal{C}) \subset \mathcal{F}$, i.e., $f^{-1}(C) \in \mathcal{F}$, for all $C \in \mathcal{C}$. We must prove that $\forall A \in \mathcal{E}$, $f^{-1}(A) \in \mathcal{F}$ (or $f^{-1}(\mathcal{E}) \subset \mathcal{F}$). Let $\mathcal{G} = \{A \in \mathcal{E} : f^{-1}(A) \in \mathcal{F}\}$ and by construction $\mathcal{C} \subset \mathcal{G}$. If $\mathcal{G}$ is a σ-algebra, then $\sigma(\mathcal{C}) = \mathcal{E} \subset \mathcal{G}$. Also, by construction $\mathcal{G} \subset \mathcal{E}$, hence $\mathcal{E} = \mathcal{G}$, which is what must be proven.

We need only show that $\mathcal{G}$ is a σ-algebra. Consider a sequence $A_1, A_2, \cdots \in \mathcal{E}$ such that $f^{-1}(A_i) \in \mathcal{F}$, i.e., $A_1, A_2 \cdots \in \mathcal{G}$. Then, since $\mathcal{E}$ is a σ-algebra, $\bigcup_{i \in \mathbb{N}} A_i \in \mathcal{E}$. And since $f^{-1}\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \bigcup_{i \in \mathbb{N}} f^{-1}(A_i)$, which is the union of elements in $\mathcal{F}$, $f^{-1}\left(\bigcup_{i \in \mathbb{N}} A_i\right) \in \mathcal{F}$.

Now, if $A \in \mathcal{E}$ is such that $f^{-1}(A) \in \mathcal{F}$, i.e., $A \in \mathcal{G}$, then $A^c \in \mathcal{E}$ and $f^{-1}(A^c) = f^{-1}(\mathbb{E}) - f^{-1}(A) = \mathbb{X} - f^{-1}(A)$ which is in $\mathcal{F}$. Hence $\mathcal{G}$ is a σ-algebra. ■

Example 3.1. *Let $\mathcal{A}_4 = \{(-\infty, a] : a \in \mathbb{R}\}$ be the collection $\mathcal{A}_4$ in Remark 1.2.4. Since $\sigma(\mathcal{A}_4) = \mathcal{B}(\mathbb{R})$,*

$$X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \sigma(\mathcal{A}_4) = \mathcal{B}(\mathbb{R}))$$

is a random variable if, and only if, $X^{-1}(\mathcal{A}_4) \subset \mathcal{F}$. Equivalently we can state X is a random variable if, and only if, $X^{-1}((-\infty, a]) = \{\omega \in \Omega : X(\omega) \leq a\} \in \mathcal{F}\ \forall\, a \in \mathbb{R}$.

The next theorem shows that continuous functions are measurable.

Theorem 3.2. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be collections of open sets associated with $\mathbb{X}_1$ and $\mathbb{X}_2$. If $f : (\mathbb{X}_1, \sigma(\mathcal{O}_1)) \to (\mathbb{X}_2, \sigma(\mathcal{O}_2))$ is continuous, then f is measurable.*

Proof. $f^{-1}(\mathcal{O}_2) \subset \mathcal{O}_1$ by continuity. But $\mathcal{O}_1 \subset \sigma(\mathcal{O}_1)$. Thus, by Theorem 3.1, f is measurable. ■

The composition of measurable functions is measurable.

Theorem 3.3. *Let $f : (\mathbb{X}, \mathcal{F}) \to (\mathbb{X}_1, \mathcal{F}_1)$ and $g : (\mathbb{X}_1, \mathcal{F}_1) \to (\mathbb{X}_2, \mathcal{F}_2)$ be measurable functions. Let $(g \circ f) : (\mathbb{X}, \mathcal{F}) \to (\mathbb{X}_2, \mathcal{F}_2)$. Then, $(g \circ f)$ is $\mathcal{F} - \mathcal{F}_2$ measurable.*

Proof. Let $F_2 \in 2^{\mathbb{X}_2}$.

$$\begin{aligned}(g \circ f)^{-1}(F_2) &= \{x \in \mathbb{X} : g(f(x)) \in F_2\} = \{x \in \mathbb{X} : f(x) \in g^{-1}(F_2)\} \\ &= \{x \in \mathbb{X} : x \in f^{-1}(g^{-1}(F_2))\}.\end{aligned}$$

If $F_2 \in \mathcal{F}_2$, and given that g is measurable, $g^{-1}(F_2) \in \mathcal{F}_1$. Since f is measurable, $f^{-1}(g^{-1}(F_2)) \in \mathcal{F}$. Hence, $(g \circ f)$ is $\mathcal{F} - \mathcal{F}_2$ measurable. ■

The next theorem shows that measurable functions can be used to transfer measures between spaces.

Theorem 3.4. *Let $(\mathbb{X}, \mathcal{F}, \mu)$ be a measure space, $(\mathbb{E}, \mathcal{E})$ be a measurable space and $f : \mathbb{X} \to \mathbb{E}$ be a $\mathcal{F} - \mathcal{E}$ measurable function. Then,*

$$m(E) := \mu(f^{-1}(E)) \text{ for all } E \in \mathcal{E}$$

is a measure on $(\mathbb{E}, \mathcal{E})$.

Proof. We verify the two defining properties of measures. First, note that if $E = \emptyset$, $m(\emptyset) = \mu(f^{-1}(\emptyset)) = \mu(\emptyset) = 0$ since μ is a measure. Second, if $\{E_n\}_{n \in \mathbb{N}}$ is a pairwise disjoint collection of sets in $\mathcal{E}$ then

$$m\left(\underset{n\in\mathbb{N}}{\cup} E_n\right) = \mu\left(f^{-1}\left(\underset{n\in\mathbb{N}}{\cup} E_n\right)\right) = \mu\left(\underset{n\in\mathbb{N}}{\cup} f^{-1}(E_n)\right) = \sum_{n\in\mathbb{N}} \mu(f^{-1}(E_n)) = \sum_{n\in\mathbb{N}} m(E_n),$$

where the next to last equality follows from the fact that μ is a measure and the last equality follows from the definition of m. ■

Example 3.2. *Let $(\mathbb{X}, \mathcal{F}, \mu) := (\Omega, \mathcal{F}, P)$, $(\mathbb{E}, \mathcal{E}) := (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and $f := X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ then $P_X(B) := P(X^{-1}(B))$ is a measure on $\mathcal{B}(\mathbb{R})$.*

The measurability of real valued functions can be characterized differently. In Example 3.1 it is shown that a function $f : (\mathbb{X}, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is said to be $\mathcal{F} - \mathcal{B}(\mathbb{R})$ measurable if for all $a \in \mathbb{R}$, the set $S_a = \{x \in \mathbb{X} : f(x) \leq a\} \in \mathcal{F}$. But since $S_a \in \mathcal{F}$ and $\mathcal{F}$ is a σ-algebra, $S_a^c \in \mathcal{F}$. Hence, f is measurable if $S_a^c = \{x \in X : f(x) > a\} \in \mathcal{F}$. Also, consider $S_{a-1/n}^c = \{x \in \mathbb{X} : f(x) > a - 1/n\}$ and let $S_a' = \bigcap_{n \in \mathbb{N}} \{x \in \mathbb{X} : f(x) > a - 1/n\} = \{x \in \mathbb{X} : f(x) \geq a\}$. Clearly, by the properties of σ-algebras $S_a' \in \mathcal{F}$. Hence, f is measurable if $\{x \in X : f(x) \geq a\} \in \mathcal{F}$. Since, $\{x \in \mathbb{X} : f(x) < a\} = \{x \in \mathbb{X} : f(x) \geq a\}^c$, measurability could also be defined in terms of $\{x \in \mathbb{X} : f(x) < a\}$.

Example 3.3. *1. Let $f : \mathbb{X} \to \mathbb{R}$, such that for all $x \in \mathbb{X}$, $f(x) = c$, $c \in \mathbb{R}$. Let $a \in \mathbb{R}$ and consider $S_a^c = \{x \in \mathbb{X} : f(x) > a\} = \{x \in \mathbb{X} : c > a\}$. If $a \geq c$, $S_a^c = \emptyset$, and if $c > a$, $S_a^c = \mathbb{X}$. Since σ-algebras always contain $\emptyset$ and $\mathbb{X}$, $f(x) = c$ is measurable.*

2. Let $E \in \mathcal{F}$ ($\mathcal{F}$ a σ-algebra). Recall that the indicator function of E is

$$I_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

If $a \geq 1$, $S_a^c = \emptyset$; if $0 \leq a < 1$, $S_a^c = E$; if $a < 0$ $S_a^c = \mathbb{X}$. Since $\mathbb{X}, \emptyset \in \mathcal{F}$ (always) and $E \in \mathcal{F}$ by construction, I_E is measurable.

3. Let $\mathbb{X} = \mathbb{R}$ and $\mathcal{F} = \mathcal{B}(\mathbb{R})$. If f is monotone increasing, i.e., $\forall\, x < x'$, $f(x) \leq f(x')$, f is measurable. Note that in this case, $S_a^c = \{x : x > y \text{ for some } y \in \mathbb{R}\} = (y, \infty)$ or $S_a^c = \{x : x \geq y\} = [y, \infty)$, which are Borel sets.

Theorem 3.5. *Let f and g be measurable real valued functions and let $c \in \mathbb{R}$. Then, $cf, f^2, f + g, fg, |f|$ are measurable.*

Proof. If $c = 0$, $cf = 0$ is a constant and consequently, measurable. If $c > 0$, then $\{x \in \mathbb{X} : cf(x) > a\} = \{x \in \mathbb{X} : f(x) > a/c\} \in \mathcal{F}$. Similarly for $c < 0$. If $a < 0$, $\{x \in \mathbb{X} : (f(x))^2 > a\} = \mathbb{X}$ and $\mathbb{X} \in \mathcal{F}$. If $a \geq 0$, $\{x \in \mathbb{X} : f^2(x) > a\} = \{x \in \mathbb{X} : f(x) > a^{1/2}$ or $f(x) < -a^{1/2}\} = \{x \in \mathbb{X} : f(x) > a^{1/2}\} \cup \{x \in \mathbb{X} : f(x) < -a^{1/2}\}$. The first set in the union is in $\mathcal{F}$ by assumption (f is measurable) and the second is in $\mathcal{F}$ by the comments following Example 3.2.

Now, $g(x) + f(x) > a \implies f(x) > a - g(x)$ which implies that there exists a rational number r such that $f(x) > r > a - g(x)$. Hence, $\{x \in \mathbb{X} : g(x) + f(x) > a\} = \underset{r\in\mathbb{Q}}{\cup}\{x \in \mathbb{X} : f(x) > r\} \cap \{x \in \mathbb{X} : g(x) > a - r\}$. Since the rational numbers are countable $\underset{r\in\mathbb{Q}}{\cup}$ is countable. Since f and g are measurable, and unions of countable measurable sets are measurable $\{x \in \mathbb{X} : g(x) + f(x) > a\} \in \mathcal{F}$. Note that $-f = (-1)f$. Hence if f is measurable, $-f$ is also measurable and so is $f + (-g) = f - g$.

Now, $fg = 1/2[(f + g)^2 - (f^2 + g^2)]$. Since $f^2, g^2, f + g, f - g$ and cf are measurable, if f, g are measurable, so is fg.

Lastly, $\{x \in \mathbb{X} : |f(x)| > a\} = \{x \in \mathbb{X} : f(x) > a \text{ or } f(x) < -a\} = \{x \in \mathbb{X} : f(x) > a\} \cup \{x \in \mathbb{X} : f(x) < -a\} = \{x \in \mathbb{X} : f(x) > a\} \cup \{x \in \mathbb{X} : -f(x) > a\}$. Since f and $-f$ are measurable, $\{x \in \mathbb{X} : |f(x)| > a\} \in \mathcal{F}$. ■

Recall that if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers

$$\liminf_{n\to\infty} x_n := \sup_{k\in\mathbb{N}} \inf_{j\geq k}\{x_j\} \text{ and } \limsup_{n\to\infty} x_n := \inf_{k\in\mathbb{N}} \sup_{j\geq k}\{x_j\}.$$

Theorem 3.6. *Let $f_i(x) : \mathbb{X} \to \mathbb{R}$ for $i = 1, 2, \cdots$ be measurable. Then $\sup\{f_1, \cdots, f_n\}$, $\inf\{f_1, \cdots, f_n\}$, $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$ and $\liminf_n f_n$ are all measurable functions.*

Proof. Let $h(x) = \sup\{f_1(x), \cdots, f_n(x)\}$. Then, $S_a = \{x \in \mathbb{X} : h(x) > a\} = \cup_{i=1}^n\{x : f_i(x) > a\}$. Consequently, since f_i is measurable, $S_a \in \mathcal{F}$. Similarly if $g(x) = \sup_{n\in\mathbb{N}} f_n(x)$, $S_a = \{x \in \mathbb{X} : g(x) > a\} = \bigcup_{n\in\mathbb{N}}\{x : f_n(x) > a\} \in \mathcal{F}$. The same argument can be made for inf. Since $\limsup_{n\to\infty} f_n = \inf_{n\geq 1} \sup_{k\geq n} f_k$, $\limsup f_n$ is measurable. The same for $\liminf_{n\to\infty} f_n$. ■

Definition 3.2. *Let $i \in I$ an arbitrary index set and $f_i : (\mathbb{X}, \mathcal{F}) \to (\mathbb{X}_i, \mathcal{F}_i)$ be $\mathcal{F} - \mathcal{F}_i$ measurable. If $\mathcal{G} \subset \mathcal{F}$ is a σ-algebra, we say that f_i is measurable with respect to $\mathcal{G}$ if $\sigma(f_i) \subset \mathcal{G}$. The smallest σ-algebra $\mathcal{G}$ that makes all f_i measurable with respect to $\mathcal{G}$ is $\sigma\left(\bigcup_{i\in I} f_i^{-1}(\mathcal{F}_i)\right)$ and is denoted by $\sigma(f_i : i \in I)$.*

3.1 Exercises

1. Suppose $(\Omega, \mathcal{F})$ and $(\mathbb{Y}, \mathcal{G})$ are measure spaces and $f : \Omega \to \mathbb{Y}$. Show that: a) $I_{f^{-1}(A)}(\omega) = (I_A \circ f)(\omega)$ for all ω; b) f is measurable if, and only if, $\sigma(\{f^{-1}(A) : A \in \mathcal{G}\}) \subset \mathcal{F}$.

2. Show that for any function $f : \mathbb{X} \to \mathbb{Y}$ and any collection of subsets $\mathcal{G}$ of $\mathbb{Y}$, $f^{-1}(\sigma(\mathcal{G})) = \sigma(f^{-1}(\mathcal{G}))$

3. Let $i \in I$ where I is an arbitrary index set. Consider $f_i : (\mathbb{X}, \mathcal{F}) \to (\mathbb{X}_i, \mathcal{F}_i)$.

    (a) Show that for all i, the smallest σ-algebra associated with $\mathbb{X}$ that makes f_i measurable is given by $f_i^{-1}(\mathcal{F}_i)$.

    (b) Show that $\sigma\left(\bigcup_{i \in I} f_i^{-1}(\mathcal{F}_i)\right)$ is the smallest σ-algebra associated with $\mathbb{X}$ that makes all f_i simultaneously measurable.

4. Let $X : (\Omega, \mathcal{F}, P) \to (S, \mathcal{B}_S)$ where $S \subset \mathbb{R}^k$ and $\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}^k\}$ be a random vector with $k \in \mathbb{N}$, and $g : (S, \mathcal{B}_S) \to (T, \mathcal{B}_T)$ be measurable where $T \subset \mathbb{R}^p$ with $p \in \mathbb{N}$. If $Y = g(X)$, show that

    (a) $\sigma(Y) := Y^{-1}(\mathcal{B}_T) \subset \sigma(X) := X^{-1}(\mathcal{B}_S)$,

    (b) if $k = p$ and g is bijective, $\sigma(Y) = \sigma(X)$.