Exercises and solutions for Fundamental Elements of Probability and Asymptotic Theory

by

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Chapter 1

Exercises

1. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a double sequence with typical value given by $f(m, n)$. Assume that

   (a) for every $n \in \mathbb{N}$, $f(m_1, n) \leq f(m_2, n)$ whenever $m_1 \leq m_2$,

   (b) for every $m \in \mathbb{N}$, $f(m, n_1) \leq f(m, n_2)$ whenever $n_1 \leq n_2$.

   Show that $\lim\limits_{n\to\infty} \left(\lim\limits_{m\to\infty} f(m, n) \right) = \lim\limits_{m\to\infty} \left(\lim\limits_{n\to\infty} f(m, n) \right) = \lim\limits_{n\to\infty} f(n, n)$.

   As a corollary, show that if $f(m, n) \geq 0$ then $\sum\limits_{n\in\mathbb{N}} \sum\limits_{m\in\mathbb{N}} f(m, n) = \sum\limits_{m\in\mathbb{N}} \sum\limits_{n\in\mathbb{N}} f(m, n)$.

   Answer: From conditions (a) and (b), $f(1, 1) \leq f(1, 2) \leq f(2, 2) \leq f(2, 3) \leq f(3, 3) \leq \cdots$ Hence, $f(m, m) \leq f(n, n)$ whenever $m \leq n$. The sequence $\{f(n, n)\}_{n\in\mathbb{N}}$ is monotonically increasing, hence it has a limit, which is either finite, if the sequence is bounded above, or infinity, if it is not. Let this limit be denoted by F. By the same reasoning, there exist limits $F_m = \lim\limits_{n\to\infty} f(m, n)$ for each $m \in \mathbb{N}$. Since $f(m, n) \leq f(n, n)$, we have that $F_m \leq F$ when $m \leq n$. Note that $F_{m_1} \leq F_{m_2}$ whenever $m_1 \leq m_2$, hence $\lim\limits_{m\to\infty} F_m = F'$ exists, and $F' \leq F$.

   To complete the proof, we need to show that $F' = F$. If F is finite, for every $\epsilon > 0$ there exists $N(\epsilon)$ such that for all $n \geq N(\epsilon)$, $F - \epsilon \leq f(n, n) \leq F$. Put $m := N(\epsilon)$, and note that

$$F_m = \lim_{n\to\infty} f(m,n) \geq f(m,m) := f(N(\epsilon), N(\epsilon)) \geq F - \epsilon.$$

Hence, $\lim_{n\to\infty} F_m = F \geq F - \epsilon$, which implies that $F \leq F'$. Combining the last inequality with $F' \leq F$ from the previous paragraph gives $F = F'$. If F is infinite, for any $C > 0$ there exists $N(C)$ such that if $n \geq N(C)$, $f(n,n) \geq C$. If $m = N(C) \leq n$ then $f(m,m) \leq f(m,n)$ and

$$C \leq f(m,m) \leq \lim_{n\to\infty} f(m,n) = F_m,$$

hence it follows that F' must be infinite.

The proof that $\lim_{n\to\infty}\left(\lim_{m\to\infty} f(m,n)\right) = \lim_{n\to\infty} f(n,n)$ follows in exactly the same way by interchanging the indexes m and n due to the symmetry of the equation.

Corollary. Let $g(p,q) = \sum_{m=1}^{p}\sum_{n=1}^{q} f(m,n)$ for $p, q \in \mathbb{N}$. Since, $f(m,n) \geq 0$, $g(p,q)$ satisfies conditions (a) and (b), establishing the result.

2. Let $\mathbb{X}$ be an arbitrary set and consider the collection of all subsets of $\mathbb{X}$ that are countable or have countable complements. Show that this collection is a σ-algebra. Use this fact to obtain the σ-algebra generated by $\mathcal{C} = \{\{x\} : x \in \mathbb{R}\}$.

   Answer: Let $\mathcal{F} = \{A \subseteq \mathbb{X} : \#A \leq \#\mathbb{N} \text{ or } \#A^c \leq \#\mathbb{N}\}$, where $\#$ indicates cardinality. First, note that $\mathbb{X} \in \mathcal{F}$ since $\mathbb{X}^c = \emptyset$, which is countable. Second, if $A \in \mathcal{F}$ then either $A = (A^c)^c$ or A^c are countable. That is, $A^c \in \mathcal{F}$. Third, if $A_n \in \mathcal{F}$ for $n \in \mathbb{N}$ we have two possible cases - A_n are all countable, or at least one of these sets is uncountable, say A_{n_0}. For the first case, $\underset{n\in\mathbb{N}}{\cup} A_n$ is the countable union of countable sets, hence it is countable and consequently in $\mathcal{F}$. For the second case, since A_{n_0} is uncountable and in $\mathcal{F}$, it must be that $A_{n_0}^c$ is countable. Also,

$$\left(\underset{n\in\mathbb{N}}{\cup} A_n\right)^c = \underset{n\in\mathbb{N}}{\cap} A_n^c \subset A_{n_0}^c.$$

Since subsets of countable sets are countable, $\left(\underset{n\in\mathbb{N}}{\cup} A_n\right)^c$ is countable, and consequently $\underset{n\in\mathbb{N}}{\cup} A_n \in \mathcal{F}$.

Now, let $\mathcal{F}$ be the σ-algebra defined above. Since $\mathcal{C} \subseteq \mathcal{F}$, $\sigma(\mathcal{C}) \subseteq \mathcal{F}$. Also, if $A \in \mathcal{F}$ either A or A^c is countable. Without loss of generality, suppose A is countable. Then, $A = \underset{x\in C}{\cup}\{x\}$ where C is a countable collection of real numbers. Hence, $A \in \sigma(\mathcal{C})$. Hence, $\mathcal{F} \subseteq \sigma(\mathcal{C})$. Combining the two set containments we have $\sigma(\mathcal{C}) = \mathcal{F}$.

3. Denote by $B(x,r)$ an open ball in $\mathbb{R}^n$ centered at x and with radius r. Show that the Borel sets are generated by the collection $B = \{B_r(x) : x \in \mathbb{R}^n,\, r > 0\}$.

   Answer: Let $B' = \{B_r(x) : x \in \mathbb{Q}^n,\, r \in \mathbb{Q}^+\}$. Then, $B' \subset B \subset \mathcal{O}_{\mathbb{R}^n}$ and $\sigma(B') \subset \sigma(B) \subset \sigma(\mathcal{O}_{\mathbb{R}^n})$.

   Now, let $S = \bigcup_{B\in B',\, B\subset O} B$. By construction $x \in S \implies x \in O$. Now, suppose $x \in O$. Then, since O is open, there exists $B(x,\epsilon)$ such that $B(x,\epsilon) \subset O$ where ϵ is a rational number. Since $\mathbb{Q}^n$ is a dense subset of $\mathbb{R}^n$, we can find $q \in \mathbb{Q}^n$ such that $\|x - q\| \leq \epsilon/2$. Consequently,

$$B(q,\epsilon/2) \subset B(x,\epsilon) \subset O.$$

   Hence, $O \subset S$. Thus, every open O can be written as $O = \bigcup_{B\in B',\, B\subset O} B$. Since B' is a collection of balls with rational radius and rational centers, B' is countable. Thus,

$$\mathcal{O}_{\mathbb{R}^n} \subset \sigma(B') \implies \sigma(\mathcal{O}_{\mathbb{R}^n}) \subset \sigma(B').$$

   Combining this set containment with $\sigma(B') \subset \sigma(B) \subset \sigma(\mathcal{O}_{\mathbb{R}^n})$ completes the proof.

4. Let $(\Omega, \mathcal{F})$ be a measurable space. Show that: a) if μ_1 and μ_2 are measures on $(\Omega, \mathcal{F})$, then $\mu_c(F) := c_1\mu_1(F) + c_2\mu_2(F)$ for $F \in \mathcal{F}$ and all $c_1, c_2 \geq 0$ is a measure; b) if $\{\mu_i\}_{i\in\mathbb{N}}$ are measures on $(\Omega, \mathcal{F})$ and $\{\alpha_i\}_{i\in\mathbb{N}}$ is a sequence of positive numbers, then $\mu_\infty(F) = \sum_{i\in\mathbb{N}} \alpha_i\mu_i(F)$ for $F \in \mathcal{F}$ is a measure.

Answer: a) First, note that $\mu_c : \mathcal{F} \to [0, \infty]$ since $c_1, c_2, \mu_1(F), \mu_2(F) \geq 0$ for all $F \in \mathcal{F}$. Second, $\mu_c(\emptyset) = c_1\mu_1(\emptyset) + c_2\mu_2(\emptyset) = 0$ since μ_1 and μ_2 are measures. Third, if $\{F_i\}_{i\in\mathbb{N}} \in \mathcal{F}$ is a pairwise disjoint collection of sets,

$$\begin{aligned}\mu_c(\cup_{i\in\mathbb{N}} F_i) &= c_1\mu_1(\cup_{i\in\mathbb{N}} F_i) + c_2\mu_2(\cup_{i\in\mathbb{N}} F_i) \\ &= c_1 \sum_{i\in\mathbb{N}} \mu_1(F_i) + c_2 \sum_{i\in\mathbb{N}} \mu_2(F_i), \text{ since } \mu_1 \text{ and } \mu_2 \text{ are measures} \\ &= \sum_{i\in\mathbb{N}} (c_1\mu_1(F_i) + c_2\mu_2(F_i)) = \sum_{i\in\mathbb{N}} \mu_c(F_i).\end{aligned}$$

b) The verification that $\mu_\infty : \mathcal{F} \to [0, \infty]$ and $\mu_\infty(\emptyset) = 0$ follows the same arguments as in item a) when examining μ_c. For σ-additivity, note that if $\{F_j\}_{j\in\mathbb{N}} \in \mathcal{F}$ is a pairwise disjoint collection of sets,

$$\mu_\infty(\cup_{j\in\mathbb{N}} F_j) = \sum_{i=1}^{\infty} \alpha_i\mu_i(\cup_{j\in\mathbb{N}} F_j) = \sum_{i=1}^{\infty} \alpha_i \sum_{j=1}^{\infty} \mu_i(F_j) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i\mu_i(F_j).$$

If we are able to interchange the sums in the last term, then we can write

$$\mu_\infty(\cup_{j\in\mathbb{N}} F_j) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \alpha_i\mu_i(F_j) = \sum_{j=1}^{\infty} \mu_\infty(F_j),$$

completing the proof. Now, note that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i\mu_i(F_j) = \lim_{n\to\infty}\lim_{m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i\mu_i(F_j) = \sup_{n\in\mathbb{N}}\sup_{m\in\mathbb{N}} \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i\mu_i(F_j) = \sup_{n\in\mathbb{N}}\sup_{m\in\mathbb{N}} S_{nm}$$

since the partial sums are increasing. Now, if $S_{nm} \in \mathbb{R}$, then

$$\sup_{n\in\mathbb{N}}\sup_{m\in\mathbb{N}} S_{nm} = \sup_{m\in\mathbb{N}}\sup_{n\in\mathbb{N}} S_{nm}.$$

Hence, to finish the proof, we require $\mu_i(F_j) < \infty$.

5. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. In this case, we call $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. Let $\nu := \mu|_{\mathcal{G}}$ be the restriction of μ to $\mathcal{G}$. That is, $\nu(G) = \mu(G)$ for all $G \in \mathcal{G}$. Is ν a measure? If μ is finite, is ν finite? If μ is a probability, is ν a probability?

Answer: Since $\emptyset \in \mathcal{G} \subset \mathcal{F}$, $\nu(\emptyset) = \mu(\emptyset) = 0$. If $\{A_i\}_{i\in\mathbb{N}} \in \mathcal{G}$ is a pairwise disjoint sequence, we have that $\{A_i\}_{i\in\mathbb{N}} \in \mathcal{F}$. Hence, $\nu(\cup_{i\in\mathbb{N}}A_i) = \mu(\cup_{i\in\mathbb{N}}A_i) = \sum_{i\in\mathbb{N}}\mu(A_i) = \sum_{i\in\mathbb{N}}\nu(A_i)$. Now, μ finite means that $\mu(\Omega) < \infty$. Since $\Omega \in \mathcal{G}$, $\nu(\Omega) = \mu(\Omega) < \infty$. The same holds for $\mu(\Omega) = 1$.

6. Show that a measure space $(\Omega, \mathcal{F}, \mu)$ is σ-finite if, and only if, there exists $\{F_n\}_{n\in\mathbb{N}} \in \mathcal{F}$ such that $\cup_{n\in\mathbb{N}}F_n = \Omega$ and $\mu(F_n) < \infty$ for all n.

   Answer: ($\Rightarrow$) By definition, $(\Omega, \mathcal{F}, \mu)$ is σ-finite if there exists and increasing sequence $A_1 \subset A_2 \subset A_3 \cdots$ such that $\cup_{n\in\mathbb{N}}A_n = \Omega$ with $\mu(A_n) < \infty$ for all n. Hence, it suffices to let $F_n = A_n$.

   ($\Leftarrow$) Let $A_n = \cup_{j=1}^{n}F_j$. Then, $A_1 \subset A_2 \subset \cdots$ and $\cup_{n\in\mathbb{N}}A_n = \cup_{j\in\mathbb{N}}F_j = \Omega$. Also, $\mu(A_n) = \mu(\cup_{j=1}^{n}F_j) \le \sum_{j=1}^{n}\mu(F_j) < \infty$ since the sum is finite and $\mu(F_j) < \infty$.

7. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$. Show that if $\sum_{n=1}^{\infty}P(E_n) < \infty$ then $P\left(\limsup_{n\to\infty} E_n\right) = 0$.

   Answer:

$$
\begin{aligned}
P\left(\limsup_{n\to\infty} E_n\right) &= P\left(\lim_{n\to\infty} \cup_{j\ge n} E_j\right) \\
&= \lim_{n\to\infty} P\left(\cup_{j\ge n}E_j\right) \text{ by continuity} \\
&\le \limsup_{n\to\infty} \sum_{j=n}^{\infty} P(E_j) \text{ by subadditivity and definition of limsup.}
\end{aligned}
$$

   Since $\sum_{n=1}^{\infty}P(E_n) < \infty$ it must be that $\sum_{j=n}^{\infty}P(E_j) \to 0$ as $n \to 0$. Consequently, $P\left(\limsup_{n\to\infty} E_n\right) = 0$.

8. Let $\{E_j\}_{j\in J}$ be a collection of pairwise disjoint events. Show that if $P(E_j) > 0$ for each $j \in J$, then J is countable.

Answer: Let $C_n = \{E_j : P(E_j) > \frac{1}{n} \text{ and } j \in J\}$. By assumption the elements of C_n are disjoint events and

$$P\left(\cup_{j_m} E_{j_m}\right) = \sum_{m=1}^{\infty} P(E_{j_m}) = \infty,$$

where the last equality follows from the fact that $P(E_{j_m}) > 0$. So, it must be that C_n has finitely many elements. Also, $\{E_j\}_{j \in J} = \cup_{n=1}^{\infty} C_n$, which is countable since it is a countable union of finite sets.

9. Consider the extended real line, i.e., $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. Let $\bar{\mathcal{B}} := \mathcal{B}(\bar{\mathbb{R}})$ be defined as the collection of sets $\bar{B}$ such that $\bar{B} = B \cup S$ where $B \in \mathcal{B}(\mathbb{R})$ and $S \in \{\emptyset, \{-\infty\}, \{\infty\}, \{-\infty, \infty\}\}$. Show that $\bar{\mathcal{B}}$ is a σ-algebra and that it is generated by a collection of sets of the form $[a, \infty]$ where $a \in \mathbb{R}$.

   Answer: Let's first show that $\bar{\mathcal{B}}$ is a σ-algebra. Since $\bar{B} = B \cup S$ with $B \in \mathcal{B}(\mathbb{R})$, we can choose $B = \mathbb{R}$ and use $S = \{-\infty, \infty\}$ to conclude that $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} \in \bar{\mathcal{B}}$. Next, note that if $\bar{B} = B \cup S$ we have that $\bar{B}^c = B^c \cap S^c$. But the complement of a set S is an element of $\{\bar{\mathbb{R}}, \mathbb{R} \cup \{\infty\}, \mathbb{R} \cup \{-\infty\}, \mathbb{R}\}$. Hence, either 1) $\bar{B}^c = B^c \cap \bar{\mathbb{R}} = B^c \cup \emptyset \in \bar{\mathcal{B}}$ or, 2) $\bar{B}^c = B^c \cap (\mathbb{R} \cup \{\infty\}) = (B^c \cap \mathbb{R}) \cup \{\infty\}$ where $B^c \cap \mathbb{R} \in \mathcal{B}$ and consequently $\bar{B}^c \in \bar{\mathcal{B}}$ or, 3) $\bar{B}^c = B^c \cap (\mathbb{R} \cup \{-\infty\}) = (B^c \cap \mathbb{R}) \cup \{-\infty\}$ where $B^c \cap \mathbb{R} \in \mathcal{B}$ and consequently $\bar{B}^c \in \bar{\mathcal{B}}$ or, 4) $\bar{B}^c = B^c \cap \mathbb{R} \in \bar{\mathcal{B}}$.

   Lastly, letting $A_i = B_i \cup S$ for $B_i \in \mathcal{B}$ we have that $\cup_{i \in \mathbb{N}} A_i = \cup_{i \in \mathbb{N}} (B_i \cup S) = (\cup_{i \in \mathbb{N}} B_i) \cup S$. Since $\cup_{i \in \mathbb{N}} B_i \in \mathcal{B}$ we have that $\cup_{i \in \mathbb{N}} A_i \in \bar{\mathcal{B}}$.

   If $\bar{\mathcal{B}}$ is a σ-algebra and $\mathcal{C} = \{[a, \infty] : a \in \mathbb{R}\}$, we need to show that $\sigma(\mathcal{C}) = \bar{\mathcal{B}}$.

   First, note that $[a, \infty] = [a, \infty) \cup \{\infty\}$ and we know that $[a, \infty) \in \mathcal{B}$. Thus, $[a, \infty] \in \bar{\mathcal{B}}$ for all $a \in \mathbb{R}$. Then, $\sigma(\mathcal{C}) \subseteq \bar{\mathcal{B}}$.

   Second, observe that for $-\infty < a \leq b < \infty$ we have $[a, b) = [a, \infty] - [b, \infty] = [a, \infty] \cap [b, \infty]^c \in \sigma(\mathcal{C})$ since $\sigma(\mathcal{C})$ contains $[a, \infty]$ and $[b, \infty]^c$ by virtue of being a

σ-algebra. Hence,

$$\mathcal{B} \subseteq \sigma(\mathcal{C}) \subseteq \bar{\mathcal{B}}.$$

Now,

$$\{\infty\} = \cap_{i\in\mathbb{N}}[i,\infty],\ \{-\infty\} = \cap_{i\in\mathbb{N}}[-\infty,-i) = \cap_{i\in\mathbb{N}}[-i,\infty]^c$$

which allows us to conclude that $\{\infty\}, \{-\infty\} \in \sigma(\mathcal{C})$. Hence, if $B \in \mathcal{B}$ all sets of the form

$$B, B\cup\{\infty\}, B\cup\{-\infty\}, B\cup\{\infty\}\cup\{-\infty\}$$

are in $\sigma(\mathcal{C})$. Hence, $\bar{\mathcal{B}} \subseteq \sigma(\mathcal{C})$. Combining this set. containment with $\sigma(\mathcal{C}) \subseteq \bar{\mathcal{B}}$ gives the result.

10. If $E_1, E_2, \cdots, E_n$ are independent events, show that the probability that none of them occur is less than or equal to $\exp\left(-\sum_{i=1}^n P(E_i)\right)$.

Answer: Let $f(x) = \exp(-x)$ and note that for $\lambda \in (0,1)$, by Taylor's Theorem

$$\exp(-x) = f(x) = f(0) + f^{(1)}(0)x + \frac{1}{2}f^{(2)}(\lambda x)x^2 = 1 - x + \frac{1}{2}\exp(-\lambda x)x^2$$

Consequently, $1 - x \leq \exp(-x)$. Now, we are interested in the event $E = \left(\cup_{i=1}^n E_i\right)^c = \cap_{i=1}^n E_i^c$. But since the $E_1, E_2, \cdots, E_n$ are independent, so is the collection $E_1^c, E_2^c, \cdots, E_n^c$. Hence, $P(E) = \prod_{i=1}^n P(E_i^c) = \prod_{i=1}^n (1-P(E_i)) \leq \prod_{i=1}^n \exp(-P(E_i)) = \exp\left(-\sum_{i=1}^n P(E_i)\right)$.

11. Let $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ be events (measurable sets) in a probability space with measure P with $\lim A_n = A$, $\lim B_n = B$, $P(B_n), P(B) > 0$ for all n. Show that $P(A_n|B) \to P(A|B)$, $P(A|B_n) \to P(A|B)$, $P(A_n|B_n) \to P(A|B)$ as $n \to \infty$.

Answer: Since $P(\cdot|B)$ is a probability measure (proved in the class notes), we have by continuity of probability measures that $P(A_n|B) \to P(A|B)$ if $\lim B_n = B$.

Now, since $\lim B_n = B$ we have that $A \cap B_n \to A \cap B$. To see this, note that if $A \cap B_n := C_n$ then $D_j = \cup_{n=j}^\infty C_n = A \cap \left(\cup_{n=1}^\infty B_n\right)$. Then, $\limsup C_n = \cap_{j=1}^\infty D_j =$

$\cap_{j=1}^{\infty}(A \cap \cup_{n=1}^{\infty} B_n) = A \cap B$. Defining lim inf for C_n we can in similar fashion that $\liminf C_n = A \cap B$. Hence, by continuity of probability measures $P(A \cap B_n) \to P(A \cap B)$ and $P(B_n) \to P(B)$. Consequently,

$$P(A|B_n) = \frac{P(A \cap B_n)}{P(B_n)} \to \frac{P(A \cap B)}{P(B)} = P(A|B).$$

Lastly, since $A_n \cap B_n \to A \cup B$, using the same arguments

$$P(A_n|B_n) = \frac{P(A_n \cap B_n)}{P(B_n)} \to \frac{P(A \cap B)}{P(B)} = P(A|B).$$

12. Let $(\mathbb{X}, \bar{\mathcal{F}}, \bar{\mu})$ be the measure space defined in Theorem 1.15 and $\mathcal{C} = \{G \in \mathbb{X} : \exists\, A,\, B \in \mathcal{F} \ni A \subset G \subset B \text{ and } \mu(B - A) = 0\}$. Show that $\bar{\mathcal{F}} = \mathcal{C}$.

Answer: $G \in \bar{\mathcal{F}} \implies G = A \cup M$ where $A \in \mathcal{F}$ and $M \in \mathcal{S}$. $M \in \mathcal{S} \implies \exists N \in \mathcal{N}_\mu \ni M \subset N$. Then,

$$A \subset G = A \cup M \subset A \cup N := B \in \mathcal{F}.$$

Now, $\mu(B - A) = \mu(B \cup A^c) = \mu((A \cup N) - A) \leq \mu(N) = 0$. Thus, $G \in \mathcal{C}$.

$G \in \mathcal{C} \implies \exists A,\, B \in \mathcal{F} \ni A \subset G \subset B$ and $\mu(B - A) = 0$. Since $A \subset G \subset B$ we have that $G - A \subset B - A$, and since $B - A$ is a μ-null set $G - A \in \mathcal{S}$. Now, $G = A \cup (G - A)$, and since $A \in \mathcal{F}$, $G \in \bar{\mathcal{F}}$.

Chapter 2

Exercises

1. Let μ be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\mu([-n,n)) < \infty$ for all $n \in \mathbb{N}$. Define,

$$F_\mu(x) := \begin{cases} \mu([0,x)) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\mu([x,0)) & \text{if } x < 0. \end{cases}$$

Show that $F_\mu : \mathbb{R} \to \mathbb{R}$ is monotonically increasing and left continuous.

Answer: Given that $\mu([-n,n)) < \infty$, F_μ takes values in $\mathbb{R}$. First, we show that all $x < x'$, $F_\mu(x) \leq F_\mu(x')$. There are three cases to be considered

(a) $(0 \leq x < x')$: if $0 < x < x'$, $F_\mu(x') - F_\mu(x) = \mu([0,x')) - \mu([0,x))$. Since $[0,x') = [0,x) \cup [x,x')$, σ-additivity of μ gives $\mu([0,x')) = \mu([0,x)) + \mu([x,x'))$ or $\mu([x,x')) = \mu([0,x')) - \mu([0,x)) = F_\mu(x') - F_\mu(x) \geq 0$. If $x = 0$, $F_\mu(x') - F_\mu(0) = \mu([0,x')) \geq 0$.

(b) $(x < 0 \leq x')$: If $x' > 0$, $F_\mu(x') - F_\mu(x) = \mu([0,x')) + \mu([x,0)) \geq 0$. If $x' = 0$, $F_\mu(0) - F_\mu(x) = \mu([x,0)) \geq 0$.

(c) $(x < x' < 0)$: $F_\mu(x') - F_\mu(x) = -\mu([x',0)) + \mu([x,0))$. Since $[x,0) = [x,x') \cup [x',0)$, σ-additivity of μ gives $\mu([x,0)) = \mu([x,x')) + \mu([x',0))$ or $\mu([x,0)) - \mu([x',0)) = F_\mu(x') - F_\mu(x) = \mu([x,x')) \geq 0$.

Second, we must show that $\lim_{n\to\infty} F_\mu(x-h_n) = F_\mu(x)$ for all $x \in \mathbb{R}$. Let $n \in \mathbb{N}$, $h_1 \geq h_2 \geq h_3 \geq \cdots$ with $h_n \downarrow 0$ as $n \to \infty$, and $h_1 > 0$. There are three cases to consider.

(a) ($x > 0$): Choose $h_1 \in (0, x)$ and define $A_n = [0, x - h_n)$. Then, $A_1 \subset A_2 \subset \cdots$ and $\lim_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n = [0, x)$. By continuity of measure from below,

$$\lim_{n\to\infty} F_\mu(x-h_n) = \lim_{n\to\infty} \mu([0, x-h_n)) = \mu([0,x)) = F_\mu(x).$$

(b) ($x = 0$): Define $A_n = [-h_n, 0)$. Then, $A_1 \supset A_2 \supset \cdots$ and $\lim_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} A_n = \emptyset$. By continuity of measures from above, and given that $\mu([-h_1, 0)) < \infty$,

$$\lim_{n\to\infty} F_\mu(-h_n) = \lim_{n\to\infty} \mu([-h_n, 0)) = \mu(\emptyset) = 0 = F_\mu(0).$$

(c) ($x < 0$): Define $A_n = [x - h_n, 0)$. Then, $A_1 \supset A_2 \supset \cdots$ and $\lim_{n\to\infty} A_n = \cap_{n=1}^\infty A_n = [x, 0)$. By continuity of measures from above and given that $\mu([x-h_1, 0)) < \infty$,

$$\lim_{n\to\infty} F_\mu(x-h_n) = \lim_{n\to\infty} -\mu([x-h_n, 0)) = -\mu([x,0)) = F_\mu(x).$$

2. Let F_μ be defined as in question 1 and let $\nu_{F_\mu}(([a,b)) = F_\mu(b) - F_\mu(a)$ for all $a \leq b$, $a, b \in \mathbb{R}$. Show that ν_{F_μ} extends uniquely to a measure on $\mathcal{B}(\mathbb{R})$ and $\nu_{F_\mu} = \mu$.

Answer: Recall that $\mathcal{S} = \{[a,b) : a \leq b,\ a, b \in \mathbb{R}\}$ is a semi-ring (if $a = b$, $[a,a) = \emptyset$). Given F_μ, we define $\nu_{F_\mu} : \mathcal{S} \to [0, \infty)$ as $\nu_{F_\mu}([a,b)) = F_\mu(b) - F_\mu(a)$ for all $a \leq b$. Since F_μ is monotonically increasing, $F_\mu(b) - F_\mu(a) \geq 0$ and $\nu_{F_\mu}([a,a) = \emptyset) = F_\mu(a) - F_\mu(a) = 0$. Also, ν_{F_μ} is finitely additive since for $a < c < b$, we have that $[a,b) = [a,c) \cup [c,b)$ and $\nu_{F_\mu}([a,b)) = F_\mu(b) - F_\mu(a) = F_\mu(c) - F_\mu(a) + F_\mu(b) - F_\mu(c) = \nu_{F_\mu}([a,c)) + \nu_{F_\mu}([c,b))$. We now show that ν_{F_μ} is σ-additive, i.e., for $[a_n, b_n)$, $n \in \mathbb{N}$ a disjoint collection such that $[a,b) = \cup_{n\in\mathbb{N}} [a_n, b_n)$, we have $\nu_{F_\mu}([a,b)) = \sum_{n\in\mathbb{N}} \nu_{F_\mu}([a_n, b_n))$. Fix ϵ_n, $\epsilon > 0$ and note that $(a_n - \epsilon_n, b_n) \supset [a_n, b_n)$. Hence, $\cup_{n\in\mathbb{N}} (a_n - \epsilon_n, b_n) \supset \cup_{n\in\mathbb{N}} [a_n, b_n) = [a,b) \supset [a, b-\epsilon]$.

Since $\underset{n\in\mathbb{N}}{\cup}(a_n-\epsilon_n,b_n)$ is an open cover for the compact set $[a,b-\epsilon]$, by the Heine-Borel Theorem, there exists $N\in\mathbb{N}$ such that

$$\cup_{n=1}^{N}\,[a_n-\epsilon_n,b_n)\supset\cup_{n=1}^{N}(a_n-\epsilon_n,b_n)\supset[a,b-\epsilon]\supset[a,b-\epsilon). \tag{2.1}$$

Now, since $\cup_{n\in\mathbb{N}}[a_n,b_n)=[a,b)$ we have $\cup_{n=1}^{N}[a_n,b_n)\subset[a,b)$ and

$$\nu_{F_\mu}([a,b))\geq\nu_{F_\mu}\left(\cup_{n=1}^{N}[a_n,b_n)\right)=\sum_{n=1}^{N}\nu_{F_\mu}\left([a_n,b_n)\right)\text{ by finite additivity.}$$

Hence, we have

$$\begin{aligned}
0&\leq\nu_{F_\mu}([a,b))-\sum_{n=1}^{N}\nu_{F_\mu}\left([a_n,b_n)\right)\\
&=\nu_{F_\mu}([a,b-\epsilon))+\nu_{F_\mu}([b-\epsilon,b))-\sum_{n=1}^{N}\big(\nu_{F_\mu}([a_n-\epsilon_n,b_n))-\nu_{F_\mu}([a_n-\epsilon_n,a_n))\big)\\
&=\nu_{F_\mu}([a,b-\epsilon))-\sum_{n=1}^{N}\nu_{F_\mu}([a_n-\epsilon_n,b_n))\text{ this term }<0\text{ by (2.1)}\\
&+\nu_{F_\mu}([b-\epsilon,b))+\sum_{n=1}^{N}\nu_{F_\mu}([a_n-\epsilon_n,a_n))\\
&\leq\nu_{F_\mu}([b-\epsilon,b))+\sum_{n=1}^{N}\nu_{F_\mu}([a_n-\epsilon_n,a_n))=F_\mu(b)-F_\mu(b-\epsilon)+\sum_{n=1}^{N}(F_\mu(a_n)-F_\mu(a_n-\epsilon_n)).
\end{aligned}$$

By left-continuity of F_μ, we can choose ϵ such that $F_\mu(b)-F_\mu(b-\epsilon)<\eta/2$ and ϵ_n such that $F_\mu(a_n)-F_\mu(a_n-\epsilon_n)<2^{-n}\,\eta/2$. Hence,

$$0\leq\nu_{F_\mu}([a,b))-\sum_{n=1}^{N}\nu_{F_\mu}\left([a_n,b_n)\right)\leq\frac{\eta}{2}\left(1+\sum_{n=1}^{N}2^{-n}\right).$$

Letting $N\to\infty$ we have that $\nu_{F_\mu}([a,b))=\sum_{n=1}^{\infty}\nu_{F_\mu}\left([a_n,b_n)\right)$.

Since ν_{F_μ} is a pre-measure on a semi-ring, by Carathéodory's Theorem, it has an extension to $\sigma(\mathcal{S})=\mathcal{B}(\mathbb{R})$. Furthermore, since for $n\in\mathbb{N}$, $[-n,n)\uparrow\mathbb{R}$ and $\nu_{F_\mu}([-n,n))=F_\mu(n)-F_\mu(-n)=\mu([0,n))+\mu([-n,0)))<\infty$, this extension is unique.

To verify that $\nu_{F_\mu}=\mu$, it suffices to verify that $\nu_{F_\mu}=\mu$ on $\mathcal{S}$, since ν_{F_μ} extends uniquely to $\mathcal{B}(\mathbb{R})$. There are three cases:

Case 1 ($0 \leq a < b$): $\nu_{F_\mu}([a,b)) = F_\mu(b) - F_\mu(a) = \mu([0,b)) - \mu([0,a)) = \mu([0,a)) + \mu([a,b)) - \mu([0,a)) = \mu([a,b))$, since $[0,b) = [0,a) \cup [a,b)$,

Case 2 ($a < 0 < b$): $\nu_{F_\mu}([a,b)) = F_\mu(b) - F_\mu(a) = \mu([0,b)) + \mu([a,0)) = \mu([a,b))$, since $[a,b) = [a,0) \cup [0,b)$,

Case 3 ($a < b \leq 0$): $\nu_{F_\mu}([a,b)) = F_\mu(b) - F_\mu(a) = -\mu([b,0)) + \mu([a,0)) = \mu([a,b))$, since $[a,b) = [a,0) - [b,0)$, which completes the proof.

3. If F is a distribution function, show that it can have an infinite number of jump discontinuities, but at most countably many.

Answer: A jump of F, denoted by $J_F(x)$ exists if $J_F(x) = F(x) - \lim_{h \to 0} F(x-h) > 0$ for $h > 0$. This happens if and only if $P(\{x\}) > 0$. Now, the collection of events $E_x := \{\{x\} : P(\{x\}) > 0\}$ is disjoint and all have positive probability. We now show that this collection is countable. Let $C_n = \{E_x : P(E_x) > \frac{1}{n} \text{ and } x \in \mathbb{R}\}$. The elements of C_n are disjoint events and

$$P\left(\cup_{x_m} E_{x_m}\right) = \sum_{m=1}^{\infty} P(E_{x_m}) = \infty,$$

where the last equality follows from the fact that $P(E_{x_m}) > 0$. So, it must be that C_n has finitely many elements. Also, $\{E_x\}_{x \in \mathbb{R}} = \cup_{n=1}^{\infty} C_n$, which is countable since it is a countable union of finite sets.

4. Show that $\lambda^1((a,b)) = b - a$ for all $a, b \in \mathbb{R}$, $a \leq b$. State and prove the same for λ^n.

Answer: Let $a < b$ and note that $[a + \frac{1}{k}, b) \uparrow (a,b)$ as $k \to \infty$. Thus, by continuity of measures,

$$\lambda((a,b)) = \lim_{k \to \infty} \lambda([a + 1/k, b) = \lim_{k \to \infty} (b - a - 1/k) = b - a.$$

Since $\lambda([a,b)) = b - a$, this proves that $\lambda(\{a\}) = 0$.

5. Consider the measure space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda^n)$. Show that for every $B \in \mathcal{B}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, $x+B \in \mathcal{B}(\mathbb{R}^n)$ and that $\lambda^n(x+B) = \lambda^n(B)$. Note: $x+B := \{z : z = x+b,\, b \in B\}$.

Answer: First, we need to show that $x + B \in \mathcal{B}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$ and for all $B \in \mathcal{B}(\mathbb{R}^n)$. Let $\mathcal{A}_x = \{B \in \mathcal{B}(\mathbb{R}^n) : x + B \in \mathcal{B}(\mathbb{R}^n)\}$ and note that $\mathcal{A}_x \subset \mathcal{B}(\mathbb{R}^n)$. Also, $\mathcal{A}_x$ is a σ-algebra associated with $\mathbb{R}^n$, since:

(a) $\mathbb{R}^n \in \mathcal{A}_x$ given that $x + b \in \mathbb{R}^n$ for all $b \in \mathbb{R}^n$ and $\mathbb{R}^n \in \mathcal{B}(\mathbb{R}^n)$,

(b) $B \in \mathcal{A}_x \implies x+B \in \mathcal{B}(\mathbb{R}^n) \implies (x+B)^c \in \mathcal{B}(\mathbb{R}^n)$. But since $(x+B)^c = x+B^c$ and $B^c \in \mathcal{B}(\mathbb{R}^n)$, $B^c \in \mathcal{A}_x$.

(c) $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}_x \implies x + A_n \in \mathcal{B}(\mathbb{R}^n)$ for all $n \in \mathbb{N}$. Since $\mathcal{B}(\mathbb{R}^n)$ is a σ-algebra $\bigcup_{n\in\mathbb{N}}(x + A_n) = x + \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{B}(\mathbb{R}^n)$. But since $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{B}(\mathbb{R}^n)$, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}_x$.

Now, let $R^{n,h} = \times_{i=1}^n [l_i, u_i) \in \mathcal{I}^{n,h} \subset \mathcal{B}(\mathbb{R}^n)$ and note that $x + R^{n,h} \in \mathcal{I}^{n,h} \subset \mathcal{B}(\mathbb{R}^n)$. Hence, $R^{n,h} \in \mathcal{A}_x \implies x + R^{n,h} \in \mathcal{A}_x$. Hence,

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{I}^{n,h}) \subset \mathcal{A}_x \subset \mathcal{B}(\mathbb{R}^n),$$

which implies that $x + B \in \mathcal{B}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$ and for all $B \in \mathcal{B}(\mathbb{R}^n)$.

Now, set $v(B) = \lambda^n(x+B)$. If $B = \emptyset$, $v(\emptyset) = \lambda^n(x+\emptyset) = \lambda^n(\emptyset) = 0$. Also, for a pairwise disjoint sequence $\{A_n\}_{n\in\mathbb{N}}$, $v\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lambda^n\left(x + \bigcup_{n\in\mathbb{N}} A_n\right) = \lambda^n\left(\bigcup_{n\in\mathbb{N}}(x + A_n)\right) = \sum_{n\in\mathbb{N}} \lambda^n(x + A_n) = \sum_{n\in\mathbb{N}} v(A_n)$. Hence, v is a measure and

$$v(R^{n,h}) = \lambda^n(x + R^{n,h}) = \prod_{i=1}^n (u_i + x_i - (l_i + x_i)) = \prod_{i=1}^n (u_i - l_i) = \lambda^n(R^{n,h}).$$

Hence, $v(R^{n,h}) = \lambda^n(R^{n,h})$ for every $R^{n,h} \in \mathcal{I}^{n,h}$. Since $\mathcal{I}^{n,h}$ is a π-system, generates $\mathcal{B}(\mathbb{R}^n)$ and admits an exhausting sequence $[-k, k) \uparrow \mathbb{R}^n$ with $\lambda^n([-k, k)^n) = (2k)^n < \infty$, we have by Carathéodory Theorem that $\lambda^n = v$ on $\mathcal{B}(\mathbb{R}^n)$.

Chapter 3

Exercises

1. Suppose $(\Omega, \mathcal{F})$ and $(\mathbb{Y}, \mathcal{G})$ are measure spaces and $f : \Omega \to \mathbb{Y}$. Show that: a) $I_{f^{-1}(A)}(\omega) = (I_A \circ f)(\omega)$ for all ω; b) f is measurable if, and only if, $\sigma(\{f^{-1}(A) : A \in \mathcal{G}\}) \subset \mathcal{F}$.

   Answer: a) For any subset $A \subset Y$, we have $f^{-1}(A) = \{\omega : f(\omega) \in A\}$. Then,

   $$I_{f^{-1}(A)}(\omega) = I_{\{\omega : f(\omega) \in A\}}(\omega) = I_A(f(\omega)) = (I_A \circ f)(\omega).$$

   b) Since f is measurable, $f^{-1}(\mathcal{G}) \subset \mathcal{F}$. By monotonicity of σ-algebras, $\sigma(f^{-1}(\mathcal{G})) = \sigma(\{f^{-1}(A) : A \in \mathcal{G}\}) \subset \mathcal{F}$. Now, $\sigma(f^{-1}(\mathcal{G})) = f^{-1}(\sigma(\mathcal{G})) = f^{-1}(\mathcal{G}) \subset \mathcal{F}$. The last set containment implies measurability.

2. Show that for any function $f : \mathbb{X} \to \mathbb{Y}$ and any collection of subsets $\mathcal{G}$ of $\mathbb{Y}$, $f^{-1}(\sigma(\mathcal{G})) = \sigma(f^{-1}(\mathcal{G}))$

   Answer: $f^{-1}(\sigma(\mathcal{G}))$ is a σ-algebra associated with $\mathbb{X}$. Since $\mathcal{G} \subset \sigma(\mathcal{G})$, $f^{-1}(\mathcal{G}) \subset f^{-1}(\sigma(\mathcal{G}))$ and consequently $\sigma(f^{-1}(\mathcal{G})) \subset f^{-1}(\sigma(\mathcal{G}))$.

   Now, as in Theorem 3.1, $\mathcal{U} = \{U \in 2^{\mathbb{Y}} : f^{-1}(U) \in \sigma(f^{-1}(\mathcal{G}))\}$ is a σ-algebra. By definition of $\mathcal{U}$

   $$f^{-1}(\mathcal{U}) \subset \sigma(f^{-1}(\mathcal{G})).$$

Also, $\mathcal{G} \subset \mathcal{U}$ since $f^{-1}(\mathcal{G}) \subset f^{-1}(\mathcal{U}) \subset \sigma(f^{-1}(\mathcal{G}))$. Since $\mathcal{U}$ is a σ-algebra we have that $\sigma(\mathcal{G}) \subset \mathcal{U}$. So,

$$f^{-1}(\sigma(\mathcal{G})) \subset f^{-1}(\mathcal{U}) \subset \sigma(f^{-1}(\mathcal{C})).$$

The last set containment combined with the reverse obtained on the last paragraph completes the proof.

3. Let $i \in I$ where I is an arbitrary index set. Consider $f_i : (\mathbb{X}, \mathcal{F}) \to (\mathbb{X}_i, \mathcal{F}_i)$.

   (a) Show that for all i, the smallest σ-algebra associated with $\mathbb{X}$ that makes f_i measurable is given by $f_i^{-1}(\mathcal{F}_i)$.

   (b) Show that $\sigma\left(\bigcup_{i \in I} f_i^{-1}(\mathcal{F}_i)\right)$ is the smallest σ-algebra associated with $\mathbb{X}$ that makes all f_i simultaneously measurable.

   Answer: a) f_i is measurable if $f_i^{-1}(\mathcal{F}_i) \subset \mathcal{F}$. But by monotonicity of $\sigma(\cdot)$ we have $\sigma(f_i^{-1}(\mathcal{F}_i)) = f_i^{-1}(\mathcal{F}_i) \subset \mathcal{F}$ since $f_i^{-1}(\mathcal{F}_i)$ is a σ-algebra. b) $f_i^{-1}(\mathcal{F}_i) \subset \mathcal{F}$ for all $i \in I$ because f_i is measurable. But any sub-σ-algebra of $\mathcal{F}$ that makes all f_i measurable functions must contain all $f_i^{-1}(\mathcal{F}_i)$, i.e., $\bigcup_{i \in I} f_i^{-1}(\mathcal{F}_i)$. However, unions of σ-algebras are not necessarily σ-algebras. Hence, we consider $\sigma\left(\bigcup_{i \in I} f_i^{-1}(\mathcal{F}_i)\right)$, the smallest σ-algebra that makes all f_i simultaneously measurable.

4. Let $X : (\Omega, \mathcal{F}, P) \to (S, \mathcal{B}_S)$ where $S \subset \mathbb{R}^k$ and $\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}^k\}$ be a random vector with $k \in \mathbb{N}$, and $g : (S, \mathcal{B}_S) \to (T, \mathcal{B}_T)$ be measurable where $T \subset \mathbb{R}^p$ with $p \in \mathbb{N}$. If $Y = g(X)$, show that

   (a) $\sigma(Y) := Y^{-1}(\mathcal{B}_T) \subset \sigma(X) := X^{-1}(\mathcal{B}_S)$,

   (b) if $k = p$ and g is bijective, $\sigma(Y) = \sigma(X)$.

Answer: (a) $E \in Y^{-1}(\mathcal{B}_T) \implies E = Y^{-1}(B_T)$ for some $B_T \in \mathcal{B}_T$. Now,

$$\begin{aligned} E &= \{\omega : Y(\omega) \in B_T\} = \{\omega : g(X(\omega)) \in B_T\} = \{\omega : X(\omega) \in g^{-1}(B_T)\} \\ &= X^{-1}(g^{-1}(B_T)). \end{aligned}$$

Since g is measurable, $g^{-1}(B_T) \in \mathcal{B}_S$ and since X is a random vector $X^{-1}(g^{-1}(B_T)) \in \sigma(X) := X^{-1}(\mathcal{B}_S)$. Hence, $\sigma(Y) \subset \sigma(X)$.

(b) First, observe that since g is bijective, it must be that $k = p$ and $S = T$. For any $B_T \in \mathcal{B}_T$,

$$\begin{aligned} g^{-1}(B_T) &= g^{-1}(g(B)) \text{ for some } B \subset S \\ &= B \in \mathcal{B}_S \text{ since } g^{-1} \text{ is an inverse function and } g \text{ is measurable.} \end{aligned}$$

Hence, any $B_T \in \mathcal{B}_T$ is such that $B_T = g(B)$ where $B \in \mathcal{B}_S$. Similarly, due to the existence of the inverse g^{-1}, for any $B_S \in \mathcal{B}_S$, $B_S = g^{-1}(B)$ where $B \in \mathcal{B}_T$. Hence, if $\mathcal{C} := \{g^{-1}(B) : B \in \mathcal{B}_T\}$ then $\mathcal{B}_S \subset \mathcal{C}$. But measurability of g assures that $\mathcal{C} \subset \mathcal{B}_S$ Hence, $X^{-1}(\mathcal{B}_S) := \sigma(X) = X^{-1}(\mathcal{C}) = \{X^{-1}(g^{-1}(B)) : B \in \mathcal{B}_T\} = \sigma(Y)$.

Chapter 4

Exercises

1. Prove Theorem 4.2.

   Answer: Let $f = \sum_{i=0}^{I} y_i I_{A_i}$ and $f = \sum_{j=0}^{J} y_j I_{B_j}$ be standard representations of f and g. Then,

   $$f \pm g = \sum_{i=0}^{I} \sum_{j=0}^{J} (y_i \pm z_j) I_{A_i \cap B_j}$$

   and

   $$fg = \sum_{i=0}^{I} \sum_{j=0}^{J} (y_i z_j) I_{A_i \cap B_j}$$

   with $(A_i \cap B_j) \cap (A_{i'} \cap B_{j'}) = \emptyset$ whenever $(i, j) \neq (i', j')$. After relabeling and merging the double sums into single sums we have the result. The case for cf is obvious. f simple implies f^+ and f^- are simple by definition, and since $|f| = f^+ + f^-$, $|f|$ is simple.

2. Show that if f is a non-negative measurable simple function, its integral, as defined in Definition 4.3 is equal to $I_\mu(f)$.

   Answer: Since f is simple and $f \leq f$, f is one of the simple functions (denoted by ϕ) appearing in Definition 21 of the class notes. Hence, $\int f d\mu \geq I_\mu(f)$. Also, if ϕ is a simple function such that $\phi \leq f$, by monotonicity of the integral of simple functions we have $I_\mu(\phi) \leq I_\mu(f)$, hence

$$\sup_{\phi} I_\mu(\phi) := \int f d\mu \leq I_\mu(f).$$

Combining the two inequalities we have $\int f d\mu = I_\mu(f)$.

3. Let $(\mathbb{X}, \mathcal{F})$ be a measurable space and $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of measures defined on it. Noting that $\mu = \sum_{n\in\mathbb{N}} \mu_n$ is also a measure on $(\mathbb{X}, \mathcal{F})$ (you don't have to prove this), show that

$$\int_{\mathbb{X}} f d\mu = \sum_{n\in\mathbb{N}} \int_{\mathbb{X}} f d\mu_n$$

for f non-negative and measurable.

Answer: First, let $f = I_F \geq 0$ for $F \in \mathcal{F}$. Then, f is measurable and

$$\int_{\mathbb{X}} f d\mu = \int_{\mathbb{X}} I_F d\mu = \mu(F) = \sum_{n\in\mathbb{N}} \mu_n(F) = \sum_{n\in\mathbb{N}} \int_{\mathbb{X}} I_F d\mu_n = \sum_{n\in\mathbb{N}} \int_{\mathbb{X}} f d\mu_n.$$

Hence, the result holds for indicator functions. Now, consider a simple non-negative function $f = \sum_{j=0}^{m} a_j I_{A_j}$ where $a_j \geq 0$ and $A_j \in \mathcal{F}$. Then,

$$\int_{\mathbb{X}} f d\mu = \int_{\mathbb{X}} \sum_{j=0}^{m} a_j I_{A_j} d\mu = \sum_{j=0}^{m} a_j \int_{\mathbb{X}} I_{A_j} d\mu = \sum_{j=0}^{m} a_j \mu(A_j) = \sum_{j=0}^{m} a_j \sum_{n\in\mathbb{N}} \mu_n(A_j)$$
$$= \sum_{n\in\mathbb{N}} \sum_{j=0}^{m} a_j \mu_n(A_j) = \sum_{n\in\mathbb{N}} \int_{\mathbb{X}} f d\mu_n.$$

Hence, the result holds for simple non-negative functions. Lastly, let f be non-negative and measurable. By Theorem 3.3 in the class notes, there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ of non-negative, non-decreasing, measurable simple function such that $\sup_{n\in\mathbb{N}} \phi_n = f$. By Beppo-Levi's Theorem

$$\int_{\mathbb{X}} f d\mu = \sup_{n\in\mathbb{N}} \int_{\mathbb{X}} \phi_n d\mu.$$

Hence,

$$\begin{aligned}
\int_{\mathbb{X}} f d\mu &= \sup_{n\in\mathbb{N}} \int_{\mathbb{X}} \phi_n d\mu = \sup_{n\in\mathbb{N}} \sum_{j=1}^{\infty} \int_{\mathbb{X}} \phi_n d\mu_j \\
&= \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} \sum_{j=1}^{m} \int_{\mathbb{X}} \phi_n d\mu_j \text{ since } \textstyle\int_{\mathbb{X}} \phi_n d\mu_j \text{ is nondecreasing.} \\
&= \sup_{m\in\mathbb{N}} \sup_{n\in\mathbb{N}} \sum_{j=1}^{m} \int_{\mathbb{X}} \phi_n d\mu_j = \sup_{m\in\mathbb{N}} \lim_{n\to\infty} \sum_{j=1}^{m} \int_{\mathbb{X}} \phi_n d\mu_j \\
&= \sup_{m\in\mathbb{N}} \sum_{j=1}^{m} \lim_{n\to\infty} \int_{\mathbb{X}} \phi_n d\mu_j \\
&= \sup_{m\in\mathbb{N}} \sum_{j=1}^{m} \int_{\mathbb{X}} \lim_{n\to\infty} \phi_n d\mu_j \text{ by Beppo-Levi's Theorem} \\
&= \sup_{m\in\mathbb{N}} \sum_{j=1}^{m} \int_{\mathbb{X}} f d\mu_j = \sum_{j\in\mathbb{N}} \int_{\mathbb{X}} f d\mu_j.
\end{aligned}$$

4. Let $(\mathbb{X}, \mathcal{F}, \mu)$ be a measure space and $f : (\mathbb{X}, \mathcal{F}, \mu) \to (\mathbb{R}, \mathcal{B})$ be measurable and non-negative. For every $F \in \mathcal{F}$ consider $\int I_F f d\mu$. Is this a measure?

   Answer: Let $v(F) = \int I_F f d\mu$. Then v is a $[0, \infty]$-valued set function defined for $F \in \mathcal{F}$. Then,

   (a) $I_\emptyset = 0$ and clearly $v(\emptyset) = 0$.

   (b) Let $F = \cup_{i\in\mathbb{N}} F_i$ be a union of pairwise disjoint sets in $\mathcal{F}$. Then, $\sum_{i=1}^{\infty} I_{F_i} = I_F$ and

   $$\begin{aligned}
   v(F) &= \int \left(\sum_{i=1}^{\infty} I_{F_i} \right) f d\mu = \int \left(\sum_{i=1}^{\infty} I_{F_i} f \right) d\mu \\
   &= \sum_{i=1}^{\infty} \int I_{F_i} f d\mu = \sum_{i=1}^{\infty} v(F_i)
   \end{aligned}$$

   .

5. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$.

   (a) Prove that $I_{\liminf_{n\to\infty} F_n} = \liminf_{n\to\infty} I_{F_n}$ and $I_{\limsup_{n\to\infty} F_n} = \limsup_{n\to\infty} I_{F_n}$.

(b) Prove that $P\left(\liminf_{n\to\infty} F_n\right) \leq \liminf_{n\to\infty} P(F_n)$.

(c) Prove that $\limsup_{n\to\infty} P(F_n) \leq P\left(\limsup_{n\to\infty} F_n\right)$.

Answer: Part (a) is straightforward by noting that $I_{\cap F_n} = \inf I_{F_n}$ and $I_{\cup F_n} = \sup I_{A_n}$. (b) Part (a) combined with Fatou's Lemma gives,

$$P(\liminf F_n) = \int I_{\liminf F_n} dP = \int \liminf I_{F_n} dP \leq \liminf \int I_{F_n} dP.$$

(c) Again, by Fatou's Lemma (the reverse) we have,

$$P(\limsup F_n) = \int I_{\limsup F_n} dP = \int \limsup I_{F_n} dP \geq \limsup \int I_{F_n} dP.$$

Chapter 5

Exercises

1. Prove Theorem 4.2.

   Answer: Let $f = \sum_{i=0}^{I} y_i I_{A_i}$ and $f = \sum_{j=0}^{J} y_j I_{B_j}$ be standard representations of f and g. Then,

   $$f \pm g = \sum_{i=0}^{I} \sum_{j=0}^{J} (y_i \pm z_j) I_{A_i \cap B_j}$$

   and

   $$fg = \sum_{i=0}^{I} \sum_{j=0}^{J} (y_i z_j) I_{A_i \cap B_j}$$

   with $(A_i \cap B_j) \cap (A_{i'} \cap B_{j'}) = \emptyset$ whenever $(i, j) \neq (i', j')$. After relabeling and merging the double sums into single sums we have the result. The case for cf is obvious. f simple implies f^+ and f^- are simple by definition, and since $|f| = f^+ + f^-$, $|f|$ is simple.

2. Prove Theorem 4.10.

   Answer: Since $f = f^+ - f^-$ and f^+ and f^- are nonnegative, use Theorems 4.6 and 4.8 in your notes.

3. Use Markov's inequality to prove the following for $a > 0$ and $g : (0, \infty) \to (0, \infty)$ that is increasing:

   $$P(|X(\omega)| \geq a) \leq \frac{1}{g(a)} \int g(|X|) dP$$

Answer: Since g is increasing, $\{\omega : |X(\omega)| \geq a\} = \{\omega : g(|X(\omega)|) \geq g(a)\}$. Hence, since g is positive

$$g(a)I_{\{\omega:|X(\omega)|\geq a\}} = g(a)I_{\{\omega:g(|X(\omega)|)\geq g(a)\}} \leq g(|X(\omega)|).$$

Integrating both sides we have $g(a)P(\{\omega : |X(\omega)| \geq a\}) \leq \int g(|X(\omega)|)dP$. This completes the proof as $g(a) > 0$.

4. Let X be a random variable defined in the probability space $(\Omega, \mathcal{F}, P)$ with $E(X^2) < \infty$. Consider a function $f : \mathbb{R} \to \mathbb{R}$. What restrictions are needed on f to guarantee that $f(X)$ is a random variable with $E(f(X)^2) < \infty$?

Answer: Recall that if $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, we say that X is a random variable (measurable real valued function) if, and only if, for all $B \in \mathcal{B}_{\mathbb{R}}$ we have $X^{-1}(B) \in \mathcal{F}$. Hence, if $h(\omega) := f(X(\omega)) = (f \circ X)(\omega) : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ we require that for all $B \in \mathcal{B}_{\mathbb{R}}$ we have $h^{-1}(B) = (f \circ X)^{-1}(B) = X^{-1}(f^{-1}(B)) \in \mathcal{F}$. That is, $f^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$.

Since X is a random variable (measurable) and given that $f^{-1}(B) \in \mathcal{B}_{\mathbb{R}}$ for all $B \in \mathcal{B}_{\mathbb{R}}$, $f(X)$ is a random variable (measurable). Since the f^2 is a continuous function of f, f^2 is also a random variable (measurable). Hence, we can consider the integrability (or not) of $f(X)^2$, i.e., whether or not $E(f(X)^2) < \infty$. We give two general restrictions on f that give $E(f(X)^2) < \infty$. First, suppose that $\sup_{\omega\in\Omega} |h(\omega)| = \sup_{\omega\in\Omega} |(f \circ X)(\omega)| < C$. Then,

$$\left| \int f^2 dP \right| \leq \int h^2 dP \leq C^2 \int dP = C^2.$$

Second, suppose that $h^2 \leq X^2$ for all $\omega \in \Omega$. Then, $\int h^2 dP \leq \int X^2 dP < \infty$.

Note that, in general, it is not true that $E(f(X)^2) < \infty$ even if $E(X^2) < \infty$. For example, suppose that $X \sim U[0, 1]$. Then, $E(X^2) = 1/3$. Now, let $Y := f(X) =$

$\tan\left(\pi(X-\frac{1}{2})\right)$ and we can easily obtain that the probability density of Y is

$$f_Y(y) = \left|\frac{d}{dy}f^{-1}(y)\right| = \left|\frac{d}{dy}\left(\frac{1}{2}+\frac{1}{\pi}\arctan(y)\right)\right| = \frac{1}{\pi}\frac{1}{1+y^2},\ y \in \mathbb{R}.$$

But this is the Cauchy density and $\int y^2 f_Y(y)dy$ does not exist.

5. Let $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B})$ be a random variable. Show that if $V(X) := E\left((X-E(X))\right)^2 = 0$ then X is a constant with probability 1.

   Answer: From your notes, if $\int_\Omega X^2 dP = 0$ then $X^2 = 0$ almost everywhere. If N is a null set $\int_\Omega X^2 dP = \int_N X^2 dP + \int_{N^c} X^2 dP = \int_N X^2 dP + \int_{N^c} 0 dP = 0$. Thus, $P(X^2 = x) = 0$ for $x \neq 0$ and $P(X^2 = 0) = 1$. But this is equivalent to $P(X = 0) = 1$. Hence, $V(X) = E\left((X-E(X))\right)^2 = 0$ implies $P(X - E(X) = 0) = P(X = E(X)) = 1$.

6. Consider the following statement:f *is continuous almost everywhere if, and only if, it is almost everywhere equal to an everywhere continuous function.* Is this true or false? Explain, with precise mathematical arguments.

   Answer: False. Consider the function $I_{\mathbb{Q}}(x)$, where $x \in \mathbb{R}$. This function is nowhere continuous in $\mathbb{R}$, but it is equal to 0 almost everywhere, an everywhere continuous function. Alternatively, the function $I_{[0,\infty)}(x)$ is continuous everywhere except at $\{0\}$, a set of measure zero. So, it is continuous almost everywhere. However, there is no everywhere continuous function in $\mathbb{R}$ that is equal $I_{[0,\infty)}(x)$ almost everywhere.

7. Adapt the proof of Lebesgue's Dominated Convergence Theorem in your notes to show that any sequence $\{f_n\}_{n\in\mathbb{N}}$ of measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ and $|f_n| \leq g$ for some g with g^p nonnegative and integrable satisfies

$$\lim_{n\to\infty}\int |f_n - f|^p d\mu = 0.$$

   Answer: (3 points) First, note that $|f_n - f|^p \leq (|f_n| + |f|)^p$. Since $|f_n - f| \to 0$ we have that $|f_n| \to |f|$. Consequently, for all $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for

$n \geq N_\epsilon$ we have

$$|f_n| - \epsilon \leq |f| \leq |f_n| + \epsilon \leq g + \epsilon$$

since $|f_n| < g$. Consequently, $|f| \leq g$, $|f|^p \leq g^p$ and $|f_n - f|^p \leq 2^p g^p$ where g^p is nonnegative and integrable. Now, letting $\phi_n = |f_n - f|^p$ we have that $\lim_{n\to\infty} \phi_n = 0$ and by Lebesgue's dominated convergence theorem in the class notes

$$\lim_{n\to\infty} \int_{\mathbb{X}} \phi_n d\mu = \int_{\mathbb{X}} \lim_{n\to\infty} \phi_n d\mu = 0.$$

8. Let λ be the one-dimensional Lebesgue measure for the Borel sets of $\mathbb{R}$. Show that for every integrable function f, the function

$$g(x) = \int_{(0,x)} f(t) d\lambda, \text{ for } x > 0$$

is continuous.

Answer: Consider a sequence $\{y_n\}_{n\in\mathbb{N}}$ with $0 < x < y_n$ such that $\lim_{n\to\infty} y_n = x$. Then,

$$\begin{aligned}
g(y_n) - g(x) &= \int_{(0,y_n)} f d\lambda - \int_{(0,x)} f d\lambda = \int_{(0,\infty)} I_{(0,y_n)} f d\lambda - \int_{(0,\infty)} I_{(0,x)} f d\lambda \\
&= \int_{(0,\infty)} (I_{(0,y_n)} - I_{(0,x)}) f d\lambda = \int_{(0,\infty)} I_{(x,y_n)} f d\lambda \\
|g(y_n) - g(x)| &\leq \int_{(0,\infty)} I_{[x,y_n)} |f| d\lambda.
\end{aligned}$$

Now, $I_{[x,y_n)}|f| \leq |f|$ and $\int_{(0,\infty)} |f| d\lambda < \infty$ since f is integrable. Also, $\lim_{n\to\infty} I_{[x,y_n)} f = 0$ almost everywhere (ae). Thus, by dominated convergence in the class notes

$$\begin{aligned}
\lim_{n\to\infty} |g(y_n) - g(x)| &\leq \lim_{n\to\infty} \int_{(0,\infty)} I_{(x,y_n)} |f| d\lambda \\
&= \int_{(0,\infty)} \lim_{n\to\infty} I_{(x,y_n)} |f| d\lambda = 0.
\end{aligned}$$

By repeating the argument for $y_n \uparrow x$ we obtain continuity of g at x.

9. Show that if X is a random variable with $E(|X|^p) < \infty$ then $|X|$ is almost everywhere real valued.

   Answer: Let $N = \{\omega : |X(\omega)| = \infty\} = \{\omega : |X(\omega)|^p = \infty\}$. Then $N = \cap_{n \in \mathbb{N}} \{\omega : |X(\omega)|^p \geq n\}$. Then,

$$\begin{aligned} P(N) &= P\left(\cap_{n \in \mathbb{N}} \{\omega : |X(\omega)|^p \geq n\}\right) \\ &= \lim_{n \to \infty} P\left(\{\omega : |X(\omega)|^p \geq n\}\right) \text{ by continuity of probability measures} \\ &\leq \lim_{n \to \infty} \frac{1}{k} \int_\Omega |X|^p dP \text{ by Markov's Inequality} \\ &= 0 \text{ since } \textstyle\int_\Omega |X|^p dP \text{ is finite.} \end{aligned}$$

10. Suppose $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B})$ is a random variable with $E(|X|) < \infty$. Let $N \in \mathcal{F}$ be such that $P(N) = 0$ and define

$$Y(\omega) = \begin{cases} X(\omega) & \text{if } \omega \notin N \\ c & \text{if } \omega \in N \end{cases},$$

   where $c \in \mathbb{R}$. Is Y integrable? Is $E(X) = E(Y)$?

   Answer: Yes, for both questions. We can change an integrable random variables at any set of measure zero without changing the integral.