Exercises and solutions for
Fundamental Elements of Probability and Asymptotic Theory

by

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Chapter 1

Exercises

1. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a double sequence with typical value given by $f(m,n)$. Assume that

   (a) for every $n \in \mathbb{N}$, $f(m_1,n) \le f(m_2,n)$ whenever $m_1 \le m_2$,

   (b) for every $m \in \mathbb{N}$, $f(m,n_1) \le f(m,n_2)$ whenever $n_1 \le n_2$.

   Show that $\lim_{n\to\infty}\left(\lim_{m\to\infty} f(m,n)\right) = \lim_{m\to\infty}\left(\lim_{n\to\infty} f(m,n)\right) = \lim_{n\to\infty} f(n,n)$.

   As a corollary, show that if $f(m,n) \ge 0$ then $\sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} f(m,n) = \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}} f(m,n)$.

   Answer: From conditions (a) and (b), $f(1,1) \le f(1,2) \le f(2,2) \le f(2,3) \le f(3,3) \le \cdots$ Hence, $f(m,m) \le f(n,n)$ whenever $m \le n$. The sequence $\{f(n,n)\}_{n\in\mathbb{N}}$ is monotonically increasing, hence it has a limit, which is either finite, if the sequence is bounded above, or infinity, if it is not. Let this limit be denoted by F. By the same reasoning, there exist limits $F_m = \lim_{n\to\infty} f(m,n)$ for each $m \in \mathbb{N}$. Since $f(m,n) \le f(n,n)$, we have that $F_m \le F$ when $m \le n$. Note that $F_{m_1} \le F_{m_2}$ whenever $m_1 \le m_2$, hence $\lim_{m\to\infty} F_m = F'$ exists, and $F' \le F$.

   To complete the proof, we need to show that $F' = F$. If F is finite, for every $\epsilon > 0$ there exists $N(\epsilon)$ such that for all $n \ge N(\epsilon)$, $F - \epsilon \le f(n,n) \le F$. Put $m := N(\epsilon)$, and note that

   $$F_m = \lim_{n\to\infty} f(m,n) \ge f(m,m) := f(N(\epsilon), N(\epsilon)) \ge F - \epsilon.$$

   Hence, $\lim_{n\to\infty} F_m = F \ge F - \epsilon$, which implies that $F \le F'$. Combining the last inequality with $F' \le F$ from the previous paragraph gives $F = F'$. If F is infinite, for any $C > 0$

there exists $N(C)$ such that if $n \geq N(C)$, $f(n,n) \geq C$. If $m = N(C) \leq n$ then $f(m,m) \leq f(m,n)$ and

$$C \leq f(m,m) \leq \lim_{n\to\infty} f(m,n) = F_m,$$

hence it follows that F' must be infinite.

The proof that $\lim_{n\to\infty}\left(\lim_{m\to\infty} f(m,n)\right) = \lim_{n\to\infty} f(n,n)$ follows in exactly the same way by interchanging the indexes m and n due to the symmetry of the equation.

Corollary. Let $g(p,q) = \sum_{m=1}^{p}\sum_{n=1}^{q} f(m,n)$ for $p,q \in \mathbb{N}$. Since, $f(m,n) \geq 0$, $g(p,q)$ satisfies conditions (a) and (b), establishing the result.

2. Let $\mathbb{X}$ be an arbitrary set and consider the collection of all subsets of $\mathbb{X}$ that are countable or have countable complements. Show that this collection is a σ-algebra. Use this fact to obtain the σ-algebra generated by $\mathcal{C} = \{\{x\} : x \in \mathbb{R}\}$.

   Answer: Let $\mathcal{F} = \{A \subseteq \mathbb{X} : \#A \leq \#\mathbb{N} \text{ or } \#A^c \leq \#\mathbb{N}\}$, where $\#$ indicates cardinality. First, note that $\mathbb{X} \in \mathcal{F}$ since $\mathbb{X}^c = \emptyset$, which is countable. Second, if $A \in \mathcal{F}$ then either $A = (A^c)^c$ or A^c are countable. That is, $A^c \in \mathcal{F}$. Third, if $A_n \in \mathcal{F}$ for $n \in \mathbb{N}$ we have two possible cases - A_n are all countable, or at least one of these sets is uncountable, say A_{n_0}. For the first case, $\underset{n\in\mathbb{N}}{\cup} A_n$ is the countable union of countable sets, hence it is countable and consequently in $\mathcal{F}$. For the second case, since A_{n_0} is uncountable and in $\mathcal{F}$, it must be that $A_{n_0}^c$ is countable. Also,

$$\left(\underset{n\in\mathbb{N}}{\cup} A_n\right)^c = \underset{n\in\mathbb{N}}{\cap} A_n^c \subset A_{n_0}^c.$$

   Since subsets of countable sets are countable, $\left(\underset{n\in\mathbb{N}}{\cup} A_n\right)^c$ is countable, and consequently $\underset{n\in\mathbb{N}}{\cup} A_n \in \mathcal{F}$.

   Now, let $\mathcal{F}$ be the σ-algebra defined above. Since $\mathcal{C} \subseteq \mathcal{F}$, $\sigma(\mathcal{C}) \subseteq \mathcal{F}$. Also, if $A \in \mathcal{F}$ either A or A^c is countable. Without loss of generality, suppose A is countable. Then, $A = \underset{x\in C}{\cup}\{x\}$ where C is a countable collection of real numbers. Hence, $A \in \sigma(\mathcal{C})$. Hence, $\mathcal{F} \subseteq \sigma(\mathcal{C})$. Combining the two set containments we have $\sigma(\mathcal{C}) = \mathcal{F}$.

3. Denote by $B(x,r)$ an open ball in $\mathbb{R}^n$ centered at x and with radius r. Show that the Borel sets are generated by the collection $B = \{B_r(x) : x \in \mathbb{R}^n,\ r > 0\}$.

   Answer: Let $B' = \{B_r(x) : x \in \mathbb{Q}^n,\ r \in \mathbb{Q}^+\}$. Then, $B' \subset B \subset \mathcal{O}_{\mathbb{R}^n}$ and $\sigma(B') \subset \sigma(B) \subset \sigma(\mathcal{O}_{\mathbb{R}^n})$.

Now, let $S = \bigcup_{B \in B',\, B \subset O} B$. By construction $x \in S \implies x \in O$. Now, suppose $x \in O$. Then, since O is open, there exists $B(x, \epsilon)$ such that $B(x, \epsilon) \subset O$ where ϵ is a rational number. Since $\mathbb{Q}^n$ is a dense subset of $\mathbb{R}^n$, we can find $q \in \mathbb{Q}^n$ such that $\|x - q\| \leq \epsilon/2$. Consequently,

$$B(q, \epsilon/2) \subset B(x, \epsilon) \subset O.$$

Hence, $O \subset S$. Thus, every open O can be written as $O = \bigcup_{B \in B',\, B \subset O} B$. Since B' is a collection of balls with rational radius and rational centers, B' is countable. Thus,

$$\mathcal{O}_{\mathbb{R}^n} \subset \sigma(B') \implies \sigma(\mathcal{O}_{\mathbb{R}^n}) \subset \sigma(B').$$

Combining this set containment with $\sigma(B') \subset \sigma(B) \subset \sigma(\mathcal{O}_{\mathbb{R}^n})$ completes the proof.

4. Let $(\Omega, \mathcal{F})$ be a measurable space. Show that: a) if μ_1 and μ_2 are measures on $(\Omega, \mathcal{F})$, then $\mu_c(F) := c_1\mu_1(F) + c_2\mu_2(F)$ for $F \in \mathcal{F}$ and all $c_1, c_2 \geq 0$ is a measure; b) if $\{\mu_i\}_{i\in\mathbb{N}}$ are measures on $(\Omega, \mathcal{F})$ and $\{\alpha_i\}_{i\in\mathbb{N}}$ is a sequence of positive numbers, then $\mu_\infty(F) = \sum_{i\in\mathbb{N}} \alpha_i\mu_i(F)$ for $F \in \mathcal{F}$ is a measure.

   Answer: a) First, note that $\mu_c : \mathcal{F} \to [0, \infty]$ since $c_1, c_2, \mu_1(F), \mu_2(F) \geq 0$ for all $F \in \mathcal{F}$. Second, $\mu_c(\emptyset) = c_1\mu_1(\emptyset) + c_2\mu_2(\emptyset) = 0$ since μ_1 and μ_2 are measures. Third, if $\{F_i\}_{i\in\mathbb{N}} \in \mathcal{F}$ is a pairwise disjoint collection of sets,

$$\begin{aligned}
\mu_c\left(\cup_{i\in\mathbb{N}} F_i\right) &= c_1\mu_1(\cup_{i\in\mathbb{N}} F_i) + c_2\mu_2(\cup_{i\in\mathbb{N}} F_i) \\
&= c_1 \sum_{i\in\mathbb{N}} \mu_1(F_i) + c_2 \sum_{i\in\mathbb{N}} \mu_2(F_i), \text{ since } \mu_1 \text{ and } \mu_2 \text{ are measures} \\
&= \sum_{i\in\mathbb{N}} (c_1\mu_1(F_i) + c_2\mu_2(F_i)) = \sum_{i\in\mathbb{N}} \mu_c(F_i).
\end{aligned}$$

   b) The verification that $\mu_\infty : \mathcal{F} \to [0, \infty]$ and $\mu_\infty(\emptyset) = 0$ follows the same arguments as in item a) when examining μ_c. For σ-additivity, note that if $\{F_j\}_{j\in\mathbb{N}} \in \mathcal{F}$ is a pairwise disjoint collection of sets,

$$\mu_\infty\left(\cup_{j\in\mathbb{N}} F_j\right) = \sum_{i=1}^{\infty} \alpha_i\mu_i\left(\cup_{j\in\mathbb{N}} F_j\right) = \sum_{i=1}^{\infty} \alpha_i \sum_{j=1}^{\infty} \mu_i\left(F_j\right) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i\mu_i\left(F_j\right).$$

   If we are able to interchange the sums in the last term, then we can write

$$\mu_\infty\left(\cup_{j\in\mathbb{N}} F_j\right) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \alpha_i\mu_i\left(F_j\right) = \sum_{j=1}^{\infty} \mu_\infty\left(F_j\right),$$

completing the proof. Now, note that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\alpha_i\mu_i\left(F_j\right)=\lim_{n\to\infty}\lim_{m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\mu_i\left(F_j\right)=\sup_{n\in\mathbb{N}}\sup_{m\in\mathbb{N}}\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\mu_i\left(F_j\right)=\sup_{n\in\mathbb{N}}\sup_{m\in\mathbb{N}}S_{nm}$$

since the partial sums are increasing. Now, if $S_{nm}\in\mathbb{R}$, then

$$\sup_{n\in\mathbb{N}}\sup_{m\in\mathbb{N}}S_{nm}=\sup_{m\in\mathbb{N}}\sup_{n\in\mathbb{N}}S_{nm}.$$

Hence, to finish the proof, we require $\mu_i(F_j)<\infty$.

5. Let $(\Omega,\mathcal{F},\mu)$ be a measure space and $\mathcal{G}\subset\mathcal{F}$ be a σ-algebra. In this case, we call $\mathcal{G}$ a sub-σ-algebra of $\mathcal{F}$. Let $\nu:=\mu|_{\mathcal{G}}$ be the restriction of μ to $\mathcal{G}$. That is, $\nu(G)=\mu(G)$ for all $G\in\mathcal{G}$. Is ν a measure? If μ is finite, is ν finite? If μ is a probability, is ν a probability?

   Answer: Since $\emptyset\in\mathcal{G}\subset\mathcal{F}$, $\nu(\emptyset)=\mu(\emptyset)=0$. If $\{A_i\}_{i\in\mathbb{N}}\in\mathcal{G}$ is a pairwise disjoint sequence, we have that $\{A_i\}_{i\in\mathbb{N}}\in\mathcal{F}$. Hence, $\nu(\cup_{i\in\mathbb{N}}A_i)=\mu(\cup_{i\in\mathbb{N}}A_i)=\sum_{i\in\mathbb{N}}\mu(A_i)=\sum_{i\in\mathbb{N}}\nu(A_i)$. Now, μ finite means that $\mu(\Omega)<\infty$. Since $\Omega\in\mathcal{G}$, $\nu(\Omega)=\mu(\Omega)<\infty$. The same holds for $\mu(\Omega)=1$.

6. Show that a measure space $(\Omega,\mathcal{F},\mu)$ is σ-finite if, and only if, there exists $\{F_n\}_{n\in\mathbb{N}}\in\mathcal{F}$ such that $\cup_{n\in\mathbb{N}}F_n=\Omega$ and $\mu(F_n)<\infty$ for all n.

   Answer: ($\Rightarrow$) By definition, $(\Omega,\mathcal{F},\mu)$ is σ-finite if there exists and increasing sequence $A_1\subset A_2\subset A_3\cdots$ such that $\cup_{n\in\mathbb{N}}A_n=\Omega$ with $\mu(A_n)<\infty$ for all n. Hence, it suffices to let $F_n=A_n$.

   ($\Leftarrow$) Let $A_n=\cup_{j=1}^{n}F_j$. Then, $A_1\subset A_2\subset\cdots$ and $\cup_{n\in\mathbb{N}}A_n=\cup_{j\in\mathbb{N}}F_j=\Omega$. Also, $\mu(A_n)=\mu(\cup_{j=1}^{n}F_j)\leq\sum_{j=1}^{n}\mu(F_j)<\infty$ since the sum is finite and $\mu(F_j)<\infty$.

7. Let $(\Omega,\mathcal{F},P)$ be a probability space and $\{E_n\}_{n\in\mathbb{N}}\subset\mathcal{F}$. Show that if $\sum_{n=1}^{\infty}P(E_n)<\infty$ then $P\left(\limsup_{n\to\infty}E_n\right)=0$.

   Answer:

$$\begin{aligned}P\left(\limsup_{n\to\infty}E_n\right)&=P\left(\lim_{n\to\infty}\cup_{j\geq n}E_j\right)\\&=\lim_{n\to\infty}P\left(\cup_{j\geq n}E_j\right)\text{ by continuity}\\&\leq\limsup_{n\to\infty}\sum_{j=n}^{\infty}P(E_j)\text{ by subadditivity and definition of limsup.}\end{aligned}$$

Since $\sum_{n=1}^{\infty} P(E_n) < \infty$ it must be that $\sum_{j=n}^{\infty} P(E_j) \to 0$ as $n \to 0$. Consequently, $P\left(\limsup_{n\to\infty} E_n\right) = 0$.

8. Let $\{E_j\}_{j\in J}$ be a collection of pairwise disjoint events. Show that if $P(E_j) > 0$ for each $j \in J$, then J is countable.

   Answer: Let $C_n = \{E_j : P(E_j) > \frac{1}{n} \text{ and } j \in J\}$. By assumption the elements of C_n are disjoint events and

$$P\left(\cup_{j_m} E_{j_m}\right) = \sum_{m=1}^{\infty} P(E_{j_m}) = \infty,$$

   where the last equality follows from the fact that $P(E_{j_m}) > 0$. So, it must be that C_n has finitely many elements. Also, $\{E_j\}_{j\in J} = \cup_{n=1}^{\infty} C_n$, which is countable since it is a countable union of finite sets.

9. Consider the extended real line, i.e., $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$. Let $\bar{\mathcal{B}} := \mathcal{B}(\bar{\mathbb{R}})$ be defined as the collection of sets $\bar{B}$ such that $\bar{B} = B \cup S$ where $B \in \mathcal{B}(\mathbb{R})$ and $S \in \{\emptyset, \{-\infty\}, \{\infty\}, \{-\infty, \infty\}\}$. Show that $\bar{\mathcal{B}}$ is a σ-algebra and that it is generated by a collection of sets of the form $[a, \infty]$ where $a \in \mathbb{R}$.

   Answer: Let's first show that $\bar{\mathcal{B}}$ is a σ-algebra. Since $\bar{B} = B \cup S$ with $B \in \mathcal{B}(\mathbb{R})$, we can choose $B = \mathbb{R}$ and use $S = \{-\infty, \infty\}$ to conclude that $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\} \in \bar{\mathcal{B}}$. Next, note that if $\bar{B} = B \cup S$ we have that $\bar{B}^c = B^c \cap S^c$. But the complement of a set S is an element of $\{\bar{\mathbb{R}}, \mathbb{R} \cup \{\infty\}, \mathbb{R} \cup \{-\infty\}, \mathbb{R}\}$. Hence, either 1) $\bar{B}^c = B^c \cap \bar{\mathbb{R}} = B^c \cup \emptyset \in \bar{\mathcal{B}}$ or, 2) $\bar{B}^c = B^c \cap (\mathbb{R} \cup \{\infty\}) = (B^c \cap \mathbb{R}) \cup \{\infty\}$ where $B^c \cap \mathbb{R} \in \mathcal{B}$ and consequently $\bar{B}^c \in \bar{\mathcal{B}}$ or, 3) $\bar{B}^c = B^c \cap (\mathbb{R} \cup \{-\infty\}) = (B^c \cap \mathbb{R}) \cup \{-\infty\}$ where $B^c \cap \mathbb{R} \in \mathcal{B}$ and consequently $\bar{B}^c \in \bar{\mathcal{B}}$ or, 4) $\bar{B}^c = B^c \cap \mathbb{R} \in \bar{\mathcal{B}}$.

   Lastly, letting $A_i = B_i \cup S$ for $B_i \in \mathcal{B}$ we have that $\cup_{i\in\mathbb{N}} A_i = \cup_{i\in\mathbb{N}} (B_i \cup S) = (\cup_{i\in\mathbb{N}} B_i) \cup S$. Since $\cup_{i\in\mathbb{N}} B_i \in \mathcal{B}$ we have that $\cup_{i\in\mathbb{N}} A_i \in \bar{\mathcal{B}}$.

   If $\bar{\mathcal{B}}$ is a σ-algebra and $\mathcal{C} = \{[a, \infty] : a \in \mathbb{R}\}$, we need to show that $\sigma(\mathcal{C}) = \bar{\mathcal{B}}$.

   First, note that $[a, \infty] = [a, \infty) \cup \{\infty\}$ and we know that $[a, \infty) \in \mathcal{B}$. Thus, $[a, \infty] \in \bar{\mathcal{B}}$ for all $a \in \mathbb{R}$. Then, $\sigma(\mathcal{C}) \subseteq \bar{\mathcal{B}}$.

   Second, observe that for $-\infty < a \le b < \infty$ we have $[a, b) = [a, \infty] - [b, \infty] = [a, \infty] \cap [b, \infty]^c \in \sigma(\mathcal{C})$ since $\sigma(\mathcal{C})$ contains $[a, \infty]$ and $[b, \infty]^c$ by virtue of being a

σ-algebra. Hence,

$$\mathcal{B} \subseteq \sigma(\mathcal{C}) \subseteq \bar{\mathcal{B}}.$$

Now,

$$\{\infty\} = \cap_{i\in\mathbb{N}}[i,\infty],\ \{-\infty\} = \cap_{i\in\mathbb{N}}[-\infty,-i) = \cap_{i\in\mathbb{N}}[-i,\infty]^c$$

which allows us to conclude that $\{\infty\},\{-\infty\} \in \sigma(\mathcal{C})$. Hence, if $B \in \mathcal{B}$ all sets of the form

$$B, B\cup\{\infty\}, B\cup\{-\infty\}, B\cup\{\infty\}\cup\{-\infty\}$$

are in $\sigma(\mathcal{C})$. Hence, $\bar{\mathcal{B}} \subseteq \sigma(\mathcal{C})$. Combining this set. containment with $\sigma(\mathcal{C}) \subseteq \bar{\mathcal{B}}$ gives the result.

10. If $E_1, E_2, \cdots, E_n$ are independent events, show that the probability that none of them occur is less than or equal to $\exp\left(-\sum_{i=1}^n P(E_i)\right)$.

    Answer: Let $f(x) = \exp(-x)$ and note that for $\lambda \in (0,1)$, by Taylor's Theorem

    $$\exp(-x) = f(x) = f(0) + f^{(1)}(0)x + \frac{1}{2}f^{(2)}(\lambda x)x^2 = 1 - x + \frac{1}{2}\exp(-\lambda x)x^2$$

    Consequently, $1 - x \leq \exp(-x)$. Now, we are interested in the event $E = (\cup_{i=1}^n E_i)^c = \cap_{i=1}^n E_i^c$. But since the $E_1, E_2, \cdots, E_n$ are independent, so is the collection $E_1^c, E_2^c, \cdots, E_n^c$. Hence, $P(E) = \prod_{i=1}^n P(E_i^c) = \prod_{i=1}^n (1-P(E_i)) \leq \prod_{i=1}^n \exp(-P(E_i)) = \exp\left(-\sum_{i=1}^n P(E_i)\right)$.

11. Let $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ be events (measurable sets) in a probability space with measure P with $\lim A_n = A$, $\lim B_n = B$, $P(B_n), P(B) > 0$ for all n. Show that $P(A_n|B) \to P(A|B)$, $P(A|B_n) \to P(A|B)$, $P(A_n|B_n) \to P(A|B)$ as $n \to \infty$.

    Answer: Since $P(\cdot|B)$ is a probability measure (proved in the class notes), we have by continuity of probability measures that $P(A_n|B) \to P(A|B)$ if $\lim B_n = B$.

    Now, since $\lim B_n = B$ we have that $A \cap B_n \to A \cap B$. To see this, note that if $A \cap B_n := C_n$ then $D_j = \cup_{n=j}^\infty C_n = A \cap (\cup_{n=1}^\infty B_n)$. Then, $\limsup C_n = \cap_{j=1}^\infty D_j = \cap_{j=1}^\infty (A \cap \cup_{n=1}^\infty B_n) = A \cap B$. Defining $\liminf$ for C_n we can in similar fashion that $\liminf C_n = A \cap B$. Hence, by continuity of probability measures $P(A\cap B_n) \to P(A\cap B)$ and $P(B_n) \to P(B)$. Consequently,

    $$P(A|B_n) = \frac{P(A\cap B_n)}{P(B_n)} \to \frac{P(A\cap B)}{P(B)} = P(A|B).$$

    Lastly, since $A_n \cap B_n \to A \cup B$, using the same arguments

    $$P(A_n|B_n) = \frac{P(A_n\cap B_n)}{P(B_n)} \to \frac{P(A\cap B)}{P(B)} = P(A|B).$$

12. Let $(\mathbb{X}, \bar{\mathcal{F}}, \bar{\mu})$ be the measure space defined in Theorem 1.15 and $\mathcal{C} = \{G \in \mathbb{X} : \exists\, A,\, B \in \mathcal{F} \ni A \subset G \subset B \text{ and } \mu(B - A) = 0\}$. Show that $\bar{\mathcal{F}} = \mathcal{C}$.

Answer: $G \in \bar{\mathcal{F}} \implies G = A \cup M$ where $A \in \mathcal{F}$ and $M \in \mathcal{S}$. $M \in \mathcal{S} \implies \exists N \in \mathcal{N}_\mu \ni M \subset N$. Then,

$$A \subset G = A \cup M \subset A \cup N := B \in \mathcal{F}.$$

Now, $\mu(B - A) = \mu(B \cup A^c) = \mu((A \cup N) - A) \leq \mu(N) = 0$. Thus, $G \in \mathcal{C}$.

$G \in \mathcal{C} \implies \exists A,\, B \in \mathcal{F} \ni A \subset G \subset B$ and $\mu(B - A) = 0$. Since $A \subset G \subset B$ we have that $G - A \subset B - A$, and since $B - A$ is a μ-null set $G - A \in \mathcal{S}$. Now, $G = A \cup (G - A)$, and since $A \in \mathcal{F}$, $G \in \bar{\mathcal{F}}$.