Chapter 2

Exercises

1. Let μ be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\mu([-n,n)) < \infty$ for all $n \in \mathbb{N}$. Define,

$$F_\mu(x) := \begin{cases} \mu([0,x)) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\mu([x,0)) & \text{if } x < 0. \end{cases}$$

Show that $F_\mu : \mathbb{R} \to \mathbb{R}$ is monotonically increasing and left continuous.

Answer: Given that $\mu([-n,n)) < \infty$, F_μ takes values in $\mathbb{R}$. First, we show that all $x < x'$, $F_\mu(x) \leq F_\mu(x')$. There are three cases to be considered

(a) $(0 \leq x < x')$: if $0 < x < x'$, $F_\mu(x') - F_\mu(x) = \mu([0,x')) - \mu([0,x))$. Since $[0,x') = [0,x) \cup [x,x')$, σ-additivity of μ gives $\mu([0,x')) = \mu([0,x)) + \mu([x,x'))$ or $\mu([x,x')) = \mu([0,x')) - \mu([0,x)) = F_\mu(x') - F_\mu(x) \geq 0$. If $x = 0$, $F_\mu(x') - F_\mu(0) = \mu([0,x')) \geq 0$.

(b) $(x < 0 \leq x')$: If $x' > 0$, $F_\mu(x') - F_\mu(x) = \mu([0,x')) + \mu([x,0)) \geq 0$. If $x' = 0$, $F_\mu(0) - F_\mu(x) = \mu([x,0)) \geq 0$.

(c) $(x < x' < 0)$: $F_\mu(x') - F_\mu(x) = -\mu([x',0)) + \mu([x,0))$. Since $[x,0) = [x,x') \cup [x',0)$, σ-additivity of μ gives $\mu([x,0)) = \mu([x,x')) + \mu([x',0))$ or $\mu([x,0)) - \mu([x',0)) = F_\mu(x') - F_\mu(x) = \mu([x,x')) \geq 0$.

Second, we must show that $\lim_{n\to\infty} F_\mu(x - h_n) = F_\mu(x)$ for all $x \in \mathbb{R}$. Let $n \in \mathbb{N}$, $h_1 \geq h_2 \geq h_3 \geq \cdots$ with $h_n \downarrow 0$ as $n \to \infty$, and $h_1 > 0$. There are three cases to consider.

(a) $(x > 0)$: Choose $h_1 \in (0, x)$ and define $A_n = [0, x - h_n)$. Then, $A_1 \subset A_2 \subset \cdots$ and $\lim_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n = [0, x)$. By continuity of measure from below,

$$\lim_{n\to\infty} F_\mu(x - h_n) = \lim_{n\to\infty} \mu([0, x - h_n)) = \mu([0, x)) = F_\mu(x).$$

(b) $(x = 0)$: Define $A_n = [-h_n, 0)$. Then, $A_1 \supset A_2 \supset \cdots$ and $\lim_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} A_n = \emptyset$. By continuity of measures from above, and given that $\mu([-h_1, 0)) < \infty$,

$$\lim_{n\to\infty} F_\mu(-h_n) = \lim_{n\to\infty} \mu([-h_n, 0)) = \mu(\emptyset) = 0 = F_\mu(0).$$

(c) $(x < 0)$: Define $A_n = [x - h_n, 0)$. Then, $A_1 \supset A_2 \supset \cdots$ and $\lim_{n\to\infty} A_n = \cap_{n=1}^{\infty} A_n = [x, 0)$. By continuity of measures from above and given that $\mu([x - h_1, 0)) < \infty$,

$$\lim_{n\to\infty} F_\mu(x - h_n) = \lim_{n\to\infty} -\mu([x - h_n, 0)) = -\mu([x, 0)) = F_\mu(x).$$

2. Let F_μ be defined as in question 1 and let $\nu_{F_\mu}(([a, b)) = F_\mu(b) - F_\mu(a)$ for all $a \leq b$, $a, b \in \mathbb{R}$. Show that ν_{F_μ} extends uniquely to a measure on $\mathcal{B}(\mathbb{R})$ and $\nu_{F_\mu} = \mu$.

Answer: Recall that $\mathcal{S} = \{[a, b) : a \leq b,\ a, b \in \mathbb{R}\}$ is a semi-ring (if $a = b$, $[a, a) = \emptyset$). Given F_μ, we define $\nu_{F_\mu} : \mathcal{S} \to [0, \infty)$ as $\nu_{F_\mu}([a, b)) = F_\mu(b) - F_\mu(a)$ for all $a \leq b$. Since F_μ is monotonically increasing, $F_\mu(b) - F_\mu(a) \geq 0$ and $\nu_{F_\mu}([a, a) = \emptyset) = F_\mu(a) - F_\mu(a) = 0$. Also, ν_{F_μ} is finitely additive since for $a < c < b$, we have that $[a, b) = [a, c) \cup [c, b)$ and $\nu_{F_\mu}([a, b)) = F_\mu(b) - F_\mu(a) = F_\mu(c) - F_\mu(a) + F_\mu(b) - F_\mu(c) = \nu_{F_\mu}([a, c)) + \nu_{F_\mu}([c, b))$. We now show that ν_{F_μ} is σ-additive, i.e., for $[a_n, b_n)$, $n \in \mathbb{N}$ a disjoint collection such that $[a, b) = \underset{n\in\mathbb{N}}{\cup} [a_n, b_n)$, we have $\nu_{F_\mu}([a, b)) = \sum_{n\in\mathbb{N}} \nu_{F_\mu}([a_n, b_n))$. Fix ϵ_n, $\epsilon > 0$ and note that $(a_n - \epsilon_n, b_n) \supset [a_n, b_n)$. Hence, $\underset{n\in\mathbb{N}}{\cup} (a_n - \epsilon_n, b_n) \supset \underset{n\in\mathbb{N}}{\cup} [a_n, b_n) = [a, b) \supset [a, b - \epsilon]$. Since $\underset{n\in\mathbb{N}}{\cup} (a_n - \epsilon_n, b_n)$ is an open cover for the compact set $[a, b - \epsilon]$, by the Heine-Borel Theorem, there exists $N \in \mathbb{N}$ such that

$$\cup_{n=1}^{N} [a_n - \epsilon_n, b_n) \supset \cup_{n=1}^{N} (a_n - \epsilon_n, b_n) \supset [a, b - \epsilon] \supset [a, b - \epsilon). \tag{2.1}$$

Now, since $\cup_{n\in\mathbb{N}} [a_n, b_n) = [a, b)$ we have $\cup_{n=1}^{N} [a_n, b_n) \subset [a, b)$ and

$$\nu_{F_\mu}([a, b)) \geq \nu_{F_\mu}\left(\cup_{n=1}^{N} [a_n, b_n)\right) = \sum_{n=1}^{N} \nu_{F_\mu}([a_n, b_n)) \text{ by finite additivity.}$$

Hence, we have

$$
\begin{aligned}
0 &\leq \nu_{F_\mu}([a,b)) - \sum_{n=1}^{N} \nu_{F_\mu}([a_n, b_n)) \\
&= \nu_{F_\mu}([a, b-\epsilon)) + \nu_{F_\mu}([b-\epsilon, b)) - \sum_{n=1}^{N} \big(\nu_{F_\mu}([a_n - \epsilon_n, b_n)) - \nu_{F_\mu}([a_n - \epsilon_n, a_n))\big) \\
&= \nu_{F_\mu}([a, b-\epsilon)) - \sum_{n=1}^{N} \nu_{F_\mu}([a_n - \epsilon_n, b_n)) \text{ this term } < 0 \text{ by (2.1)} \\
&+ \nu_{F_\mu}([b-\epsilon, b)) + \sum_{n=1}^{N} \nu_{F_\mu}([a_n - \epsilon_n, a_n)) \\
&\leq \nu_{F_\mu}([b-\epsilon, b)) + \sum_{n=1}^{N} \nu_{F_\mu}([a_n - \epsilon_n, a_n)) = F_\mu(b) - F_\mu(b-\epsilon) + \sum_{n=1}^{N} (F_\mu(a_n) - F_\mu(a_n - \epsilon_n)).
\end{aligned}
$$

By left-continuity of F_μ, we can choose ϵ such that $F_\mu(b) - F_\mu(b-\epsilon) < \eta/2$ and ϵ_n such that $F_\mu(a_n) - F_\mu(a_n - \epsilon_n) < 2^{-n}\,\eta/2$. Hence,

$$
0 \leq \nu_{F_\mu}([a,b)) - \sum_{n=1}^{N} \nu_{F_\mu}([a_n, b_n)) \leq \frac{\eta}{2}\left(1 + \sum_{n=1}^{N} 2^{-n}\right).
$$

Letting $N \to \infty$ we have that $\nu_{F_\mu}([a,b)) = \sum_{n=1}^{\infty} \nu_{F_\mu}([a_n, b_n))$.

Since ν_{F_μ} is a pre-measure on a semi-ring, by Carathéodory's Theorem, it has an extension to $\sigma(\mathcal{S}) = \mathcal{B}(\mathbb{R})$. Furthermore, since for $n \in \mathbb{N}$, $[-n, n) \uparrow \mathbb{R}$ and $\nu_{F_\mu}([-n, n)) = F_\mu(n) - F_\mu(-n) = \mu([0, n)) + \mu([-n, 0))) < \infty$, this extension is unique.

To verify that $\nu_{F_\mu} = \mu$, it suffices to verify that $\nu_{F_\mu} = \mu$ on $\mathcal{S}$, since ν_{F_μ} extends uniquely to $\mathcal{B}(\mathbb{R})$. There are three cases:

Case 1 ($0 \leq a < b$): $\nu_{F_\mu}([a,b)) = F_\mu(b) - F_\mu(a) = \mu([0,b)) - \mu([0,a)) = \mu([0,a)) + \mu([a,b)) - \mu([0,a)) = \mu([a,b))$, since $[0,b) = [0,a) \cup [a,b)$,

Case 2 ($a < 0 < b$): $\nu_{F_\mu}([a,b)) = F_\mu(b) - F_\mu(a) = \mu([0,b)) + \mu([a,0)) = \mu([a,b))$, since $[a,b) = [a,0) \cup [0,b)$,

Case 3 ($a < b \leq 0$): $\nu_{F_\mu}([a,b)) = F_\mu(b) - F_\mu(a) = -\mu([b,0)) + \mu([a,0)) = \mu([a,b))$, since $[a,b) = [a,0) - [b,0)$, which completes the proof.

3. If F is a distribution function, show that it can have an infinite number of jump discontinuities, but at most countably many.

Answer: A jump of F, denoted by $J_F(x)$ exists if $J_F(x) = F(x) - \lim_{h\to 0} F(x-h) > 0$ for $h > 0$. This happens if and only if $P(\{x\}) > 0$. Now, the collection of events $E_x := \{\{x\} : P(\{x\}) > 0\}$ is disjoint and all have positive probability. We now show that this collection is countable. Let $C_n = \{E_x : P(E_x) > \frac{1}{n} \text{ and } x \in \mathbb{R}\}$. The elements of C_n are disjoint events and

$$P\left(\cup_{x_m} E_{x_m}\right) = \sum_{m=1}^{\infty} P(E_{x_m}) = \infty,$$

where the last equality follows from the fact that $P(E_{x_m}) > 0$. So, it must be that C_n has finitely many elements. Also, $\{E_x\}_{x\in\mathbb{R}} = \cup_{n=1}^{\infty} C_n$, which is countable since it is a countable union of finite sets.

4. Show that $\lambda^1((a,b)) = b - a$ for all $a, b \in \mathbb{R}$, $a \leq b$. State and prove the same for λ^n.

   Answer: Let $a < b$ and note that $[a + \frac{1}{k}, b) \uparrow (a, b)$ as $k \to \infty$. Thus, by continuity of measures,

   $$\lambda((a,b)) = \lim_{k\to\infty} \lambda([a + 1/k, b) = \lim_{k\to\infty} (b - a - 1/k) = b - a.$$

   Since $\lambda([a,b)) = b - a$, this proves that $\lambda(\{a\}) = 0$.

5. Consider the measure space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda^n)$. Show that for every $B \in \mathcal{B}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, $x + B \in \mathcal{B}(\mathbb{R}^n)$ and that $\lambda^n(x + B) = \lambda^n(B)$. Note: $x + B := \{z : z = x + b,\ b \in B\}$.

   Answer: First, we need to show that $x + B \in \mathcal{B}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$ and for all $B \in \mathcal{B}(\mathbb{R}^n)$. Let $\mathcal{A}_x = \{B \in \mathcal{B}(\mathbb{R}^n) : x + B \in \mathcal{B}(\mathbb{R}^n)\}$ and note that $\mathcal{A}_x \subset \mathcal{B}(\mathbb{R}^n)$. Also, $\mathcal{A}_x$ is a σ-algebra associated with $\mathbb{R}^n$, since:

   (a) $\mathbb{R}^n \in \mathcal{A}_x$ given that $x + b \in \mathbb{R}^n$ for all $b \in \mathbb{R}^n$ and $\mathbb{R}^n \in \mathcal{B}(\mathbb{R}^n)$,

   (b) $B \in \mathcal{A}_x \implies x + B \in \mathcal{B}(\mathbb{R}^n) \implies (x + B)^c \in \mathcal{B}(\mathbb{R}^n)$. But since $(x + B)^c = x + B^c$ and $B^c \in \mathcal{B}(\mathbb{R}^n)$, $B^c \in \mathcal{A}_x$.

   (c) $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A}_x \implies x + A_n \in \mathcal{B}(\mathbb{R}^n)$ for all $n \in \mathbb{N}$. Since $\mathcal{B}(\mathbb{R}^n)$ is a σ-algebra $\bigcup_{n\in\mathbb{N}}(x + A_n) = x + \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{B}(\mathbb{R}^n)$. But since $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{B}(\mathbb{R}^n)$, $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}_x$.

   Now, let $R^{n,h} = \times_{i=1}^{n} [l_i, u_i) \in \mathcal{I}^{n,h} \subset \mathcal{B}(\mathbb{R}^n)$ and note that $x + R^{n,h} \in \mathcal{I}^{n,h} \subset \mathcal{B}(\mathbb{R}^n)$. Hence, $R^{n,h} \in \mathcal{A}_x \implies x + R^{n,h} \in \mathcal{A}_x$. Hence,

   $$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{I}^{n,h}) \subset \mathcal{A}_x \subset \mathcal{B}(\mathbb{R}^n),$$

which implies that $x + B \in \mathcal{B}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$ and for all $B \in \mathcal{B}(\mathbb{R}^n)$.

Now, set $v(B) = \lambda^n(x+B)$. If $B = \emptyset$, $v(\emptyset) = \lambda^n(x+\emptyset) = \lambda^n(\emptyset) = 0$. Also, for a pairwise disjoint sequence $\{A_n\}_{n\in\mathbb{N}}$, $v\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lambda^n\left(x + \bigcup_{n\in\mathbb{N}} A_n\right) = \lambda^n\left(\bigcup_{n\in\mathbb{N}}(x + A_n)\right) = \sum_{n\in\mathbb{N}} \lambda^n(x + A_n) = \sum_{n\in\mathbb{N}} v(A_n)$. Hence, v is a measure and

$$v(R^{n,h}) = \lambda^n(x + R^{n,h}) = \prod_{i=1}^{n}(u_i + x_i - (l_i + x_i)) = \prod_{i=1}^{n}(u_i - l_i) = \lambda^n(R^{n,h}).$$

Hence, $v(R^{n,h}) = \lambda^n(R^{n,h})$ for every $R^{n,h} \in \mathcal{I}^{n,h}$. Since $\mathcal{I}^{n,h}$ is a π-system, generates $\mathcal{B}(\mathbb{R}^n)$ and admits an exhausting sequence $[-k, k) \uparrow \mathbb{R}^n$ with $\lambda^n([-k, k)^n) = (2k)^n < \infty$, we have by Carathéodory Theorem that $\lambda^n = v$ on $\mathcal{B}(\mathbb{R}^n)$.