Chapter 3

Exercises

1. Suppose $(\Omega, \mathcal{F})$ and $(\mathbb{Y}, \mathcal{G})$ are measure spaces and $f : \Omega \to \mathbb{Y}$. Show that: a) $I_{f^{-1}(A)}(\omega) = (I_A \circ f)(\omega)$ for all ω; b) f is measurable if, and only if, $\sigma(\{f^{-1}(A) : A \in \mathcal{G}\}) \subset \mathcal{F}$.

   Answer: a) For any subset $A \subset Y$, we have $f^{-1}(A) = \{\omega : f(\omega) \in A\}$. Then,

   $$I_{f^{-1}(A)}(\omega) = I_{\{\omega: f(\omega)\in A\}}(\omega) = I_A(f(\omega)) = (I_A \circ f)(\omega).$$

   b) Since f is measurable, $f^{-1}(\mathcal{G}) \subset \mathcal{F}$. By monotonicity of σ-algebras, $\sigma(f^{-1}(\mathcal{G})) = \sigma(\{f^{-1}(A) : A \in \mathcal{G}\}) \subset \mathcal{F}$. Now, $\sigma(f^{-1}(\mathcal{G})) = f^{-1}(\sigma(\mathcal{G})) = f^{-1}(\mathcal{G}) \subset \mathcal{F}$. The last set containment implies measurability.

2. Show that for any function $f : \mathbb{X} \to \mathbb{Y}$ and any collection of subsets $\mathcal{G}$ of $\mathbb{Y}$, $f^{-1}(\sigma(\mathcal{G})) = \sigma(f^{-1}(\mathcal{G}))$

   Answer: $f^{-1}(\sigma(\mathcal{G}))$ is a σ-algebra associated with $\mathbb{X}$. Since $\mathcal{G} \subset \sigma(\mathcal{G})$, $f^{-1}(\mathcal{G}) \subset f^{-1}(\sigma(\mathcal{G}))$ and consequently $\sigma(f^{-1}(\mathcal{G})) \subset f^{-1}(\sigma(\mathcal{G}))$.

   Now, as in Theorem 3.1, $\mathcal{U} = \{U \in 2^{\mathbb{Y}} : f^{-1}(U) \in \sigma(f^{-1}(\mathcal{G}))\}$ is a σ-algebra. By definition of $\mathcal{U}$

   $$f^{-1}(\mathcal{U}) \subset \sigma(f^{-1}(\mathcal{G})).$$

   Also, $\mathcal{G} \subset \mathcal{U}$ since $f^{-1}(\mathcal{G}) \subset f^{-1}(\mathcal{U}) \subset \sigma(f^{-1}(\mathcal{G}))$. Since $\mathcal{U}$ is a σ-algebra we have that $\sigma(\mathcal{G}) \subset \mathcal{U}$. So,

   $$f^{-1}(\sigma(\mathcal{G})) \subset f^{-1}(\mathcal{U}) \subset \sigma(f^{-1}(\mathcal{C})).$$

   The last set containment combined with the reverse obtained on the last paragraph completes the proof.

3. Let $i \in I$ where I is an arbitrary index set. Consider $f_i : (\mathbb{X}, \mathcal{F}) \to (\mathbb{X}_i, \mathcal{F}_i)$.

   (a) Show that for all i, the smallest σ-algebra associated with $\mathbb{X}$ that makes f_i measurable is given by $f_i^{-1}(\mathcal{F}_i)$.

   (b) Show that $\sigma\left(\bigcup_{i\in I} f_i^{-1}(\mathcal{F}_i)\right)$ is the smallest σ-algebra associated with $\mathbb{X}$ that makes all f_i simultaneously measurable.

   Answer: a) f_i is measurable if $f_i^{-1}(\mathcal{F}_i) \subset \mathcal{F}$. But by monotonicity of $\sigma(\cdot)$ we have $\sigma(f_i^{-1}(\mathcal{F}_i)) = f_i^{-1}(\mathcal{F}_i) \subset \mathcal{F}$ since $f_i^{-1}(\mathcal{F}_i)$ is a σ-algebra. b) $f_i^{-1}(\mathcal{F}_i) \subset \mathcal{F}$ for all $i \in I$ because f_i is measurable. But any sub-σ-algebra of $\mathcal{F}$ that makes all f_i measurable functions must contain all $f_i^{-1}(\mathcal{F}_i)$, i.e., $\bigcup_{i\in I} f_i^{-1}(\mathcal{F}_i)$. However, unions of σ-algebras are not necessarily σ-algebras. Hence, we consider $\sigma\left(\bigcup_{i\in I} f_i^{-1}(\mathcal{F}_i)\right)$, the smallest σ-algebra that makes all f_i simultaneously measurable.

4. Let $X : (\Omega, \mathcal{F}, P) \to (S, \mathcal{B}_S)$ where $S \subset \mathbb{R}^k$ and $\mathcal{B}_S = \{B \cap S : B \in \mathcal{B}^k\}$ be a random vector with $k \in \mathbb{N}$, and $g : (S, \mathcal{B}_S) \to (T, \mathcal{B}_T)$ be measurable where $T \subset \mathbb{R}^p$ with $p \in \mathbb{N}$. If $Y = g(X)$, show that

   (a) $\sigma(Y) := Y^{-1}(\mathcal{B}_T) \subset \sigma(X) := X^{-1}(\mathcal{B}_S)$,

   (b) if $k = p$ and g is bijective, $\sigma(Y) = \sigma(X)$.

   Answer: (a) $E \in Y^{-1}(\mathcal{B}_T) \implies E = Y^{-1}(B_T)$ for some $B_T \in \mathcal{B}_T$. Now,

$$
\begin{aligned}
E &= \{\omega : Y(\omega) \in B_T\} = \{\omega : g(X(\omega)) \in B_T\} = \{\omega : X(\omega) \in g^{-1}(B_T)\} \\
&= X^{-1}(g^{-1}(B_T)).
\end{aligned}
$$

   Since g is measurable, $g^{-1}(B_T) \in \mathcal{B}_S$ and since X is a random vector $X^{-1}(g^{-1}(B_T)) \in \sigma(X) := X^{-1}(\mathcal{B}_S)$. Hence, $\sigma(Y) \subset \sigma(X)$.

   (b) First, observe that since g is bijective, it must be that $k = p$ and $S = T$. For any $B_T \in \mathcal{B}_T$,

$$
\begin{aligned}
g^{-1}(B_T) &= g^{-1}(g(B)) \text{ for some } B \subset S \\
&= B \in \mathcal{B}_S \text{ since } g^{-1} \text{ is an inverse function and } g \text{ is measurable.}
\end{aligned}
$$

   Hence, any $B_T \in \mathcal{B}_T$ is such that $B_T = g(B)$ where $B \in \mathcal{B}_S$. Similarly, due to the existence of the inverse g^{-1}, for any $B_S \in \mathcal{B}_S$, $B_S = g^{-1}(B)$ where $B \in \mathcal{B}_T$. Hence, if

$\mathcal{C} := \{g^{-1}(B) : B \in \mathcal{B}_T\}$ then $\mathcal{B}_S \subset \mathcal{C}$. But measurability of g assures that $\mathcal{C} \subset \mathcal{B}_S$ Hence, $X^{-1}(\mathcal{B}_S) := \sigma(X) = X^{-1}(\mathcal{C}) = \{X^{-1}(g^{-1}(B)) : B \in \mathcal{B}_T\} = \sigma(Y)$.