Chapter 4

Exercises

1. Prove Theorem 4.2.

   Answer: Let $f = \sum_{i=0}^{I} y_i I_{A_i}$ and $f = \sum_{j=0}^{J} y_j I_{B_j}$ be standard representations of f and g. Then,

   $$f \pm g = \sum_{i=0}^{I} \sum_{j=0}^{J} (y_i \pm z_j) I_{A_i \cap B_j}$$

   and

   $$fg = \sum_{i=0}^{I} \sum_{j=0}^{J} (y_i z_j) I_{A_i \cap B_j}$$

   with $(A_i \cap B_j) \cap (A_{i'} \cap B_{j'}) = \emptyset$ whenever $(i, j) \neq (i', j')$. After relabeling and merging the double sums into single sums we have the result. The case for cf is obvious. f simple implies f^+ and f^- are simple by definition, and since $|f| = f^+ + f^-$, $|f|$ is simple.

2. Show that if f is a non-negative measurable simple function, its integral, as defined in Definition 4.3 is equal to $I_\mu(f)$.

   Answer: Since f is simple and $f \leq f$, f is one of the simple functions (denoted by ϕ) appearing in Definition 21 of the class notes. Hence, $\int f d\mu \geq I_\mu(f)$. Also, if ϕ is a simple function such that $\phi \leq f$, by monotonicity of the integral of simple functions we have $I_\mu(\phi) \leq I_\mu(f)$, hence

   $$\sup_\phi I_\mu(\phi) := \int f d\mu \leq I_\mu(f).$$

   Combining the two inequalities we have $\int f d\mu = I_\mu(f)$.

3. Let $(\mathbb{X}, \mathcal{F})$ be a measurable space and $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of measures defined on it. Noting that $\mu = \sum_{n\in\mathbb{N}} \mu_n$ is also a measure on $(\mathbb{X}, \mathcal{F})$ (you don't have to prove this), show that

$$\int_{\mathbb{X}} f d\mu = \sum_{n\in\mathbb{N}} \int_{\mathbb{X}} f d\mu_n$$

for f non-negative and measurable.

Answer: First, let $f = I_F \geq 0$ for $F \in \mathcal{F}$. Then, f is measurable and

$$\int_{\mathbb{X}} f d\mu = \int_{\mathbb{X}} I_F d\mu = \mu(F) = \sum_{n\in\mathbb{N}} \mu_n(F) = \sum_{n\in\mathbb{N}} \int_{\mathbb{X}} I_F d\mu_n = \sum_{n\in\mathbb{N}} \int_{\mathbb{X}} f d\mu_n.$$

Hence, the result holds for indicator functions. Now, consider a simple non-negative function $f = \sum_{j=0}^{m} a_j I_{A_j}$ where $a_j \geq 0$ and $A_j \in \mathcal{F}$. Then,

$$\begin{aligned}\int_{\mathbb{X}} f d\mu &= \int_{\mathbb{X}} \sum_{j=0}^{m} a_j I_{A_j} d\mu = \sum_{j=0}^{m} a_j \int_{\mathbb{X}} I_{A_j} d\mu = \sum_{j=0}^{m} a_j \mu(A_j) = \sum_{j=0}^{m} a_j \sum_{n\in\mathbb{N}} \mu_n(A_j) \\ &= \sum_{n\in\mathbb{N}} \sum_{j=0}^{m} a_j \mu_n(A_j) = \sum_{n\in\mathbb{N}} \int_{\mathbb{X}} f d\mu_n.\end{aligned}$$

Hence, the result holds for simple non-negative functions. Lastly, let f be non-negative and measurable. By Theorem 3.3 in the class notes, there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ of non-negative, non-decreasing, measurable simple function such that $\sup_{n\in\mathbb{N}} \phi_n = f$. By Beppo-Levi's Theorem

$$\int_{\mathbb{X}} f d\mu = \sup_{n\in\mathbb{N}} \int_{\mathbb{X}} \phi_n d\mu.$$

Hence,

$$
\begin{aligned}
\int_{\mathbb{X}} f d\mu &= \sup_{n\in\mathbb{N}} \int_{\mathbb{X}} \phi_n d\mu = \sup_{n\in\mathbb{N}} \sum_{j=1}^{\infty} \int_{\mathbb{X}} \phi_n d\mu_j \\
&= \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} \sum_{j=1}^{m} \int_{\mathbb{X}} \phi_n d\mu_j \text{ since } \textstyle\int_{\mathbb{X}} \phi_n d\mu_j \text{ is nondecreasing.} \\
&= \sup_{m\in\mathbb{N}} \sup_{n\in\mathbb{N}} \sum_{j=1}^{m} \int_{\mathbb{X}} \phi_n d\mu_j = \sup_{m\in\mathbb{N}} \lim_{n\to\infty} \sum_{j=1}^{m} \int_{\mathbb{X}} \phi_n d\mu_j \\
&= \sup_{m\in\mathbb{N}} \sum_{j=1}^{m} \lim_{n\to\infty} \int_{\mathbb{X}} \phi_n d\mu_j \\
&= \sup_{m\in\mathbb{N}} \sum_{j=1}^{m} \int_{\mathbb{X}} \lim_{n\to\infty} \phi_n d\mu_j \text{ by Beppo-Levi's Theorem} \\
&= \sup_{m\in\mathbb{N}} \sum_{j=1}^{m} \int_{\mathbb{X}} f d\mu_j = \sum_{j\in\mathbb{N}} \int_{\mathbb{X}} f d\mu_j.
\end{aligned}
$$

4. Let $(\mathbb{X}, \mathcal{F}, \mu)$ be a measure space and $f : (\mathbb{X}, \mathcal{F}, \mu) \to (\mathbb{R}, \mathcal{B})$ be measurable and non-negative. For every $F \in \mathcal{F}$ consider $\int I_F f d\mu$. Is this a measure?
Answer: Let $v(F) = \int I_F f d\mu$. Then v is a $[0, \infty]$-valued set function defined for $F \in \mathcal{F}$. Then,

   (a) $I_\emptyset = 0$ and clearly $v(\emptyset) = 0$.

   (b) Let $F = \cup_{i\in\mathbb{N}} F_i$ be a union of pairwise disjoint sets in $\mathcal{F}$. Then, $\sum_{i=1}^{\infty} I_{F_i} = I_F$ and

$$
\begin{aligned}
v(F) &= \int \left(\sum_{i=1}^{\infty} I_{F_i} \right) f d\mu = \int \left(\sum_{i=1}^{\infty} I_{F_i} f \right) d\mu \\
&= \sum_{i=1}^{\infty} \int I_{F_i} f d\mu = \sum_{i=1}^{\infty} v(F_i)
\end{aligned}
$$

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5. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$.

   (a) Prove that $I_{\liminf_{n\to\infty} F_n} = \liminf_{n\to\infty} I_{F_n}$ and $I_{\limsup_{n\to\infty} F_n} = \limsup_{n\to\infty} I_{F_n}$.

   (b) Prove that $P\left(\liminf_{n\to\infty} F_n\right) \leq \liminf_{n\to\infty} P(F_n)$.

(c) Prove that $\limsup_{n\to\infty} P(F_n) \leq P\left(\limsup_{n\to\infty} F_n\right)$.

Answer: Part (a) is straightforward by noting that $I_{\cap F_n} = \inf I_{F_n}$ and $I_{\cup F_n} = \sup I_{A_n}$. (b) Part (a) combined with Fatou's Lemma gives,

$$P(\liminf F_n) = \int I_{\liminf F_n} dP = \int \liminf I_{F_n} dP \leq \liminf \int I_{F_n} dP.$$

(c) Again, by Fatou's Lemma (the reverse) we have,

$$P(\limsup F_n) = \int I_{\limsup F_n} dP = \int \limsup I_{F_n} dP \geq \limsup \int I_{F_n} dP.$$