

Chapter 6

Exercises

1. Let $X_1, X_2 \in \mathcal{L}^2$ and define $\text{Cov}(X_1, X_2) = E([X_1 - E(X_1)][X_2 - E(X_2)])$. Show that $\text{Cov}(X_1, X_2) = E(X_1 X_2) - E(X_1)E(X_2)$ and that if X_1 and X_2 are independent $\text{Cov}(X_1, X_2) = 0$.

Answer: From the definition of $\text{Cov}(X_1, X_2)$ and linearity of expectations

$$\text{Cov}(X_1, X_2) = E(X_1 X_2 - X_1 E(X_2) - X_2 E(X_1) + E(X_1)E(X_2)) = E(X_1 X_2) - E(X_1)E(X_2).$$

Independence of X_1 and X_2 implies that $E(X_1 X_2) = E(X_1)E(X_2)$. Hence, $\text{Cov}(X_1, X_2) = 0$.

2. Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of random variables that are independent and share the same continuous distribution. Let p be a permutation of $\{1, \dots, n\}$ for $n \in \mathbb{N}$. Show that $(X_1, \dots, X_n)$ and $(X_{p(1)}, \dots, X_{p(n)})$ have the same distribution.

Answer: Since the random variables are independent and have the same distribution, say F ,

$$P(X_1 \leq x_1, \dots, X_n \leq x_n) = \prod_{i=1}^n F(x_i)$$

for all $x_i \in \mathbb{R}$. If $\{p(i)\}_{i=1}^n$ is a permutation of $\{1, \dots, n\}$, then

$$P(X_{p(1)} \leq x_1, \dots, X_{p(n)} \leq x_n) = \prod_{i=1}^n F(x_i).$$

Hence, $(X_1, \dots, X_n)$ and $X_{p(1)}, \dots, X_{p(n)}$ have the same distribution.

3. Let I be a finite index set and consider the collection of σ -algebras $\{\mathcal{B}_i\}_{i \in I}$. Show that this collection is independent if, and only if, for every choice of non-negative $\mathcal{B}_i$ -measurable random variable X_i , we have $E\left(\prod_{i \in I} X_i\right) = \prod_{i \in I} E(X_i)$.

Answer: If $E\left(\prod_{i \in I} X_i\right) = \prod_{i \in I} E(X_i)$ whenever $X_i \in \mathcal{B}_i$, then for any $A_i \in \mathcal{B}_i$, take $X_i = I_{A_i}$ and

$$E\left(\prod_{i \in I} X_i\right) = P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i) = \prod_{i \in I} E(X_i)$$

and consequently $\{A_i\}_{i \in I}$ are independent and $\{\mathcal{B}_i\}_{i \in I}$ are independent σ -algebras.

Now, suppose $\{\mathcal{B}_i\}_{i \in I}$ are independent σ -algebras. For $A_i \in \mathcal{B}_i$ and $X_i = I_{A_i}$ we have

$$E\left(\prod_{i \in I} X_i\right) = \prod_{i \in I} E(X_i).$$

Now, if $\{X_i\}_{i \in I}$ are simple, then write $X_i = \sum_j x_{ij} I_{A_{ij}}$ for $A_{ij} \in \mathcal{B}_i$. Then, we have

$$\begin{aligned} E\left(\prod_{i \in I} X_i\right) &= E\left(\prod_{i \in I} \sum_{j(i)} x_{ij(i)} I_{A_{ij(i)}}\right) = E\left(\sum_{j(i), i \in I} \prod_{i \in I} x_{ij(i)} I_{\bigcap_{i \in I} A_{ij(i)}}\right) \\ &= \sum_{j(i), i \in I} \prod_{i \in I} x_{ij(i)} P\left(\bigcap_{i \in I} A_{ij(i)}\right) = \sum_{j(i), i \in I} \prod_{i \in I} x_{ij(i)} P(A_{ij(i)}) \\ &= \prod_{i \in I} \sum_{j(i)} x_{ij(i)} P(A_{ij(i)}) = \prod_{i \in I} E(X_i). \end{aligned}$$

If X_i is a non-negative $\mathcal{B}_i$ -measurable function, there exists $X_i^{(n)}$ such that $X_i^{(n)}$ is simple and $0 < X_i^{(n)} \uparrow X_i$. Then, it follows that $\prod_{i \in I} X_i^{(n)} \uparrow \prod_{i \in I} X_i$ and by the monotone convergence theorem $E\left(\prod_{i \in I} X_i^{(n)}\right) \uparrow E\left(\prod_{i \in I} X_i\right)$ and from the previous argument, the left side is $\prod_{i \in I} E\left(X_i^{(n)}\right) \uparrow \prod_{i \in I} E(X_i)$ again using the monotone convergence theorem.

4. If E is an event that is independent of the π -system P and $E \in \sigma(P)$, then $P(E)$ is either 0 or 1.

Answer: Set $\mathcal{C}_1 = E$ and $\mathcal{C}_2 = P$ and it follows that $\mathcal{C}_1$ is independent of $\mathcal{C}_2$. This implies that $\sigma(\mathcal{C}_1)$ is independent of $\sigma(\mathcal{C}_2)$. Therefore, E is independent of E and $P(E) = 0$ or 1 .

5. Let $\{A_i\}_{i=1}^n$ be independent events. Show that $P(\cup_{i=1}^n A_i) = 1 - \prod_{i=1}^n P(A_i^c)$.

Answer: By De Morgan's Law $(\cup_{i=1}^n A_i)^c = \cap_{i=1}^n A_i^c$. Hence,

$$P\left(\left(\bigcup_{i=1}^n A_i\right)^c\right) = 1 - P\left(\bigcup_{i=1}^n A_i\right) = P\left(\bigcap_{i=1}^n A_i^c\right)$$

which implies

$$P\left(\bigcup_{i=1}^n A_i\right) = 1 - P\left(\bigcap_{i=1}^n A_i^c\right) = 1 - \prod_{i=1}^n P(A_i^c)$$

where the last equality follows from the fact that if $\{A_i\}_{i=1}^n$ are independent events, so are $\{A_i^c\}_{i=1}^n$.

6. We have proved that if X and Y are independent, then $f(X)$ and $g(Y)$ are independent if f and g are measurable. Is it possible to have X and Y be dependent and $f(X)$ and $g(Y)$ be independent? If so, give an example, if not, prove.

Answer: Yes, it is possible. Consider two independent random variables X_1 and X_2 and another random variable W that is independent of X_1 and X_2 and takes on the values 1 and -1 with probability $1/2$ each. Now, define two new random variables $X = WX_1$ and $Y = WX_2$. X and Y are functionally connected and cannot be independent. However, $X^2 = X_1^2$ and $Y^2 = X_2^2$, which are independent since X_1 and X_2 are independent.