Exercises for Math Camp - Set 1

by

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1. Prove Theorem 1 in your notes.

2. Prove item 2 of Theorem 3 on your class notes.

3. Let I_A be the indicator function of a set $A \subseteq \mathbb{X}$. If $A_i \subseteq \mathbb{X}$ and $i \in I$ where I is an arbitrary index set. Prove the following for any two $i, j \in I$:

   (a) $I_{A_i \cap A_j} = I_{A_i} I_{A_j}$,

   (b) $I_{A_i \cup A_j} = \min\{I_{A_i} + I_{A_j}, 1\}$,

   (c) $I_{A_i - A_j} = I_{A_i} - I_{A_i \cap A_j}$,

   (d) $I_{A_i \cup A_j} = I_{A_i} + I_{A_j} - I_{A_i \cap A_j}$,

   (e) $I_{A_i \cup A_j} = \max\{I_{A_i}, I_{A_j}\}$,

   (f) $I_{A_i \cap A_j} = \min\{I_{A_i}, I_{A_j}\}$.

4. Use the principle of induction to prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

5. Show that $(A - B) \cap C = (A \cap C) - (B \cap C)$.

6. Show that $A \cup B = A$ and $A \cap B = A$ imply $A = B$

7. let $i \in I$ where I is an arbitrary index set. Show that $\bigcup\limits_{i \in I} A_i - \bigcup\limits_{i \in I} B_i \subseteq \bigcup\limits_{i \in I} (A_i - B_i)$.

8. Let $f : \mathbb{X} \to \mathbb{Y}$ be a function and consider $A_1, A_2 \subseteq \mathbb{X}$ and $B_1, B_2 \subseteq \mathbb{Y}$. Show that,

$$f^{-1}(B_1^c) = (f^{-1}(B_1))^c \text{ and}$$

$$f(A_1^c) = (f(A_1))^c \text{ and } f(A_1 - A_2) = f(A_1) - f(A_2) \text{ if } f \text{ is bijective.}$$

9. Let $f : \mathbb{X} \to \mathbb{Y}$ be a function and consider $A \subseteq \mathbb{X}$ and $B \subseteq \mathbb{Y}$. Show that, $f(f^{-1}(B)) \subseteq B$, $A \subseteq f^{-1}(f(A))$.

10. Show the equality on item 3 of Remark 2 in your notes.

11. In Theorem 13 of your notes, show that f in the proof is strictly increasing in $(0, 1)$.

12. Let $\{A_i\}_{i \in \mathbb{N}}$ be a sequence of sets of cardinality $\mathcal{C}$. Show that the cardinality of $\bigcup_{i \in \mathbb{N}} A_i = \mathcal{C}$

13. Let V be a real vector space and $v_1, v_2, \cdots, v_n \in V$. Let W be the collection of all linear combinations of $v_1, v_2, \cdots, v_n$. Show that W is a subspace of V.

14. Show that the following subset of $\mathbb{R}^2$ is a subspace: all (x, y) such that $x + 4y = 0$.

15. Show that the following subsets of $\mathbb{R}^3$ are subspaces: b.1) all (x, y, z) such that $x + y + z = 0$; b.2) all (x, y, z) such that $x = y$ and $2y = z$.

16. Let $x = \{x_1, x_2, \cdots\}$ be a sequence of real numbers. We say that a sequence is bounded if there exists a constant $M \in \mathbb{R}$ such that $|x_k| < M$ for all k. Define another sequence $y = \{y_1, y_2, \cdots\}$ and $x + y = \{x_1 + y_1, x_2 + y_2, \cdots\}$ and $ax = \{ax_1, ax_2, \cdots\}$ for $a \in \mathbb{R}$. Show that the set of all bounded sequences is a vector space.

17. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a double sequence with typical value given by $f(m, n)$. Assume that

    (a) for every $n \in \mathbb{N}$, $f(m_1, n) \leq f(m_2, n)$ whenever $m_1 \leq m_2$,

    (b) for every $m \in \mathbb{N}$, $f(m, n_1) \leq f(m, n_2)$ whenever $n_1 \leq n_2$.

    Show that $\lim\limits_{n\to\infty} \left(\lim\limits_{m\to\infty} f(m, n) \right) = \lim\limits_{m\to\infty} \left(\lim\limits_{n\to\infty} f(m, n) \right) = \lim\limits_{n\to\infty} f(n, n)$.
    As a corollary, show that if $f(m, n) \geq 0$ then $\sum\limits_{n\in\mathbb{N}} \sum\limits_{m\in\mathbb{N}} f(m, n) = \sum\limits_{m\in\mathbb{N}} \sum\limits_{n\in\mathbb{N}} f(m, n)$.

18. Prove items 4, 5 and 6 in Theorem 21 of your notes.

19. Show that if S is a convex set in a normed vector space, its interior and its closure are convex.

20. Let $(\mathbb{X}, \|\cdot\|)$ be a normed vector space. Show that any norm $\|\cdot\|$ is a continuous function.

21. Let $V = \{x : \text{such that } x = (x_1, \cdots, x_n, 0, \cdots) \text{ and } n \in \mathbb{N}\}$ with norm $\|x\| = \max_{1\leq i\leq n} |x_i|$. Let $f(x) = \sum_{i=1}^{n} i x_i$. Show that f is linear but is not bounded.

22. If $f, g \in L(\mathbb{X}, \mathbb{Y})$ show that $f \circ g \in L(\mathbb{X}, \mathbb{Y})$.

23. Show that the set of all bounded sequences is a vector space.

24. Let $\mathcal{C}$ be any collection of subsets of $\mathbb{X}$. Show that there exists a smallest topology that contains $\mathcal{C}$.