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On functional form representation of multi-output production technologies

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Abstract The introduction of directional distance functions has given researchers an alternative to Shephard distance functions. In this paper we conduct a Monte Carlo study to investigate the performance of distance functions as an approximation for models of technology. Our results indicate that quadratic representations of technology have better approximation properties than translog parameterizations.

Keywords Distance functions · Parameterization

JEL Classification D24 · C63

Distance functions play a central role in modeling production technologies, creating indexes and productivity measures, as well as in duality theory. With the introduction of shortage functions (Luenberger 1992) or, as they have also been called, directional distance functions (Chambers et al. 1998), researchers have been given alternatives to the Shephard input and output distance functions (Shephard 1953, 1970). Both sets of functions fully represent a technology but they differ in that directional distance functions meet the translation property, whereas the Shephard distance functions are homogeneous of degree one. This difference has a profound impact on what parameterization may be chosen.

In this paper we discuss various parametric forms that belong to the family of generalized quadratic functions.[1] These functions are linear in parameters and quadratic. Combining them with the translation or homogeneity property yields parametric representations, such as the quadratic and the translog function. We conduct a Monte Carlo study as a means to suggest the 'best' distance function to approximate a production technology. Our simulations suggest that the output set is better parameterized *via* a quadratic output directional distance function than with a translog Shephard output distance function.

Other papers that investigate the properties of various parametric functional forms include Guilkey et al. (1983), Perroni and Rutherford (1998), Vardanyan and Noh (2006) and Färe et al. (2008). Guilkey et al. examined the behavior of three functions that are used to model cost functions. They set up a Monte Carlo experiment and estimate cost functions to show that the translog functional form, although not perfect, is acceptable, as it outperforms other parameterizations, such as the generalized Leontief functional form (Diewert 1971). Perroni and Rutherford (1998) studied the global properties of the cost function by calibrating its parameters to satisfy a particular structure of the

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[1] Chambers (1988) terminology. Diewert (2002) terms them "transformed quadratic functions" and Färe and Sung (1986) illustrate that these functions have a second-order Taylor series approximation.

production technology. Their results indicate that the translog functional form often violates regularity conditions and note that better behaved parametric specifications, such as the nonseparable nested constant elasticity of substitution function (Perroni and Rutherford 1995), are available.[2] Vardanyan and Noh (2006) used data from the U.S. electric utility industry to show that the translog function has poor global approximation properties relative to the quadratic function when parameterizing the production technologies that are associated with the production of socially undesirable outputs. Finally, more recently the comparison between the translog and quadratic functions was also carried out in the context of consumer choice theory (Färe et al. 2008). The authors used a Monte Carlo simulation to compare the performance of translog expenditure function to that of quadratic benefit function and found that quadratic parameterizations perform better based on the fraction of monotonicity and curvature violations.

1 Functional forms

Let $x \in \Re_+^K$ represent inputs used to produce outputs $y \in \Re_+^M$ and $T = \left\{(x, y) \in \Re_+^{K+M} : x \, \text{can produce} \, y\right\}$ denote the production technology. For all $(x,y) \in T$ this technology can also be represented by the output sets

$$P(x) = \{y : (x, y) \in T\}, \quad x \in \Re_+^K. \tag{1}$$

We assume that $P(x)$ satisfies standard axioms (Färe and Primont 1995), which include compactness and free disposability of inputs and outputs. The Shephard output distance function is defined in terms of the output sets $P(x)$ as

$$D(x, y) = \inf\{\varphi > 0 : y/\varphi \in P(x)\}. \tag{2}$$

The two properties of this function that are important for this paper are, namely:

(i) *Homogeneity:* $D(x,\mu y) = \mu D(x,y)$, $\mu > 0$,
(ii) *Representation:* $D(x,y) \leq 1$ *if and only if* $y \in P(x)$.

The homogeneity property follows from the definition of the distance function, and the representation is due to the assumption of free disposability of outputs. See Färe and Primont (1995) or Shephard (1970) for further discussion of this distance function.

To define the directional output distance function, a directional vector $g \in \Re_+^M$, $g \neq 0$ is required. This vector determines the direction in which the production frontier is approached. In case of Shephard it is the output vector itself, i.e. $g = y$. Thus, the directional distance function is defined as

$$\vec{D}(x, y; g) = \sup\{\psi > 0 : (y + \psi g) \in P(x)\}. \tag{3}$$

This function meets:

iii) *Translation:* $\vec{D}(x, y + \alpha g; g) = \vec{D}(x, y; g) - \alpha, \ \alpha \in \Re$,
iv) *Representation:* $\vec{D}(x, y; g) \geq 0$ *if and only if* $y \in P(x)$.

The translation property follows from the definition of the function and the representation property is due to the assumption of strong disposability of outputs. See Färe and Grosskopf (2004) for more on the properties of directional output distance functions.

Both distance functions meet representation and are therefore dual to the revenue function,

$$R(x, p) = \max\{py : y \in P(x)\}, \tag{4}$$

where $p \in \Re_+^M$ denotes output prices. If $g = y$ then

$$\vec{D}(x, y; g) = \frac{1}{D(x, y)} - 1. \tag{5}$$

To generate parametric functional forms for the two types of distance functions we will restrict ourselves to the class of functions which are linear in parameters.[3] This class has been referred to as the class of transformed quadratic functions (Diewert 2002), or the functions that can be represented as second-order Taylor series approximations (Färe and Sung 1986). For any $q \in \Re^I$ with component q_i, and twice differentiable functions $h : \Re \to \Re$ and $\rho : \Re \to \Re$ with an inverse ρ^{-1}, a function in this class can be written as

$$F(q) = \rho^{-1}\left(\alpha_0 + \sum\nolimits_{i=1}^{I} a_i h(q_i) + \sum\nolimits_{i=1}^{I} \sum\nolimits_{i'=1}^{I} a_{ii'} h(q_i) h(q_{i'})\right). \tag{6}$$

Färe and Sung (1986)[4] solve for functions that are simultaneously transformed quadratic and homogeneous of degree one, i.e. functions that meet (6) and i), and find two solutions: the translog function (Christensen et al. 1971),

$$F(q) = a_0 + \sum\nolimits_{i=1}^{I} a_i \ln q_i + \sum\nolimits_{i=1}^{I} \sum\nolimits_{i'=1}^{I} a_{ii'} \ln q_i \ln q_{i'} \tag{7}$$

and the quadratic mean of order r function,

$$F(q) = \left(\sum\nolimits_{i=1}^{I} \sum\nolimits_{i'=1}^{I} a_{ii'} q_i^{r/2} q_{i'}^{r/2}\right)^{1/r}. \tag{8}$$

[2] Note that the generalized Leontief function does not belong to the family of generalized quadratic functions and is therefore not considered here.

[3] There are two reasons why the functions that are nonlinear in parameters have been excluded from our analysis. First, linear-in-parameters specifications are the most common functional forms used in economics. Second, to our knowledge the functional equations that are associated with nonlinear-in-parameters functions that satisfy homogeneity or translation have not yet been solved.

[4] The authors assume $I = 2$.

Färe and Lundberg (2005)[5] solve for functions that are simultaneously transformed quadratic and satisfy the translation property, i.e. functions that meet (6) and iii), and find the only two solutions: the quadratic function,[6]

$$F(q) = a_0 + \sum_{i=1}^{I} a_i q_i + \sum_{i=1}^{I} \sum_{i'=1}^{I} a_{ii'} q_i q_{i'} \tag{9}$$

and

$$F(q) = \frac{1}{2\lambda} \ln \sum_{i=1}^{I} \sum_{i'=1}^{I} a_{ii'} \exp(\lambda q_i) \exp(\lambda q_{i'}). \tag{10}$$

The translog and the quadratic functions are the only ones we will compare in this paper. These functions have both first- and second-order terms, while the other two functions have second-order terms only. It is likely that the quality of approximation of the true technology attained by specifications (7) and (9) will be different, thus a reliable benchmark is needed. We investigate the quality of these approximations by designing a Monte Carlo experiment that allows us to shed light on the relative strengths and weaknesses of these two models, as well as to pinpoint the factors that can affect the quality of approximation for each of them. We compare the translog and the quadratic functions with respect to the functions themselves, their first-order derivatives (shadow prices), and their second-order derivatives (Morishima (1967) elasticities of substitution).

2 Monte Carlo experiments

In our simulation study we consider two sets of true technologies. The first set consists of three representations of polynomial-of-order-four technologies, whereas the second set includes three versions of translog-of-order-four technologies. We assume that two inputs produce two outputs, i.e. $K = M = 2$. The polynomial technology is given by

$$P^Q(x) = \{(y_1, y_2) : y_2 = \beta_0^Q + \beta_1^Q y_1 + \beta_2^Q y_1^2 + \beta_3^Q y_1^3 + \beta_4^Q y_1^4 + x_1^{0.9} x_2^{0.8} \equiv f^Q(y_1, x)\},\ x \in \Re_+^2. \tag{11}$$

The parameter vector $\beta^Q = (\beta_0^Q, \beta_1^Q, \beta_2^Q, \beta_3^Q, \beta_4^Q)$ models the varying rate of change in the opportunity cost of one output in terms of the other, or the marginal rate of transformation. It is chosen in the following way:

	Model Q1	Model Q2	Model Q3
β_0^Q	10.70	10.10	9.60
β_1^Q	−0.91	−0.72	−0.54
β_2^Q	0.50×10^{-5}	0.50×10^{-4}	0.10×10^{-2}
β_3^Q	0.10×10^{-4}	0.10×10^{-3}	0.10×10^{-2}
β_4^Q	-0.45×10^{-3}	-0.12×10^{-2}	-0.24×10^{-2}

The corresponding plots of the output set boundaries across the valid range of y_1 and assuming $x_1 = x_2 = 1$ are given in Panel I of Fig. 1. This choice of parameters is primarily motivated by the curvature requirements that must be met with regards to the output set frontiers, i.e. their concavity. Note that Model Q1 assumes the lowest rate of change in the marginal rate of transformation, whereas Model Q3 is the "most concave" of the three.[7]

We generate the quantities y_1 by randomly drawing sample sizes (N) of 50, 100, 500, and 1,000 observations from a gamma distribution characterized by the density function $p(y_1) = y_1^{\lambda-1} e^{-y_1/\theta} \left(\Gamma(\lambda)\theta^{\lambda}\right)^{-1}$, where $\Gamma(\cdot)$ is the gamma function, $(\lambda, \theta) \in \Re_+^2$. For each of three true models we consider two different values for (λ, θ):

Type-A Models-$y_1^A \sim \text{Gamma}(\lambda = 5, \theta = 0.5)$,

Type-B Models-$y_1^B \sim \text{Gamma}(\lambda = 18, \theta = 0.25)$.

The input quantities are drawn from the uniform distribution as $x_k \sim \text{Uniform}(0, 1)$, $k = 1, 2$. Finally, the values of the second output are obtained in the following way:

$$y_2 = f^Q(y_1, x) - \upsilon, \tag{12}$$

where υ is an exponentially distributed random noise that captures technical inefficiency with the density function $p(\upsilon) = \exp\{-\upsilon\}$. Note that type-A and type-B models differ by the area of $P^Q(x)$ around which the output quantities are clustered. While in type-A models the quantity of the second output is generally greater than that of the first output for the majority of observations in the sample, type-B models are associated with more balanced quantities of y_2 and y_1. Figure 2 illustrates the scatter plots for just two samples sizes and both of these models assuming $x_1 = x_2 = 1$.

[5] The authors assume that $\alpha_0 = 0$ in (6).

[6] These two functions were suggested for the parameterization of the directional distance function by Chambers (1998) who took $\lambda = 1/2$ in (10).

[7] During the initial stages of our analysis we experimented with a number of other forms of the true technology, including polynomials of orders both higher and lower than four. The outcomes of these trials were always in line with ones we summarize here for a more limited number of true models. Hence, the decision to include only a small number of true technologies does not result in a significant loss of generality and it allow us to keep the simulation results easily interpretable—a welcomed characteristic in any Monte Carlo study.

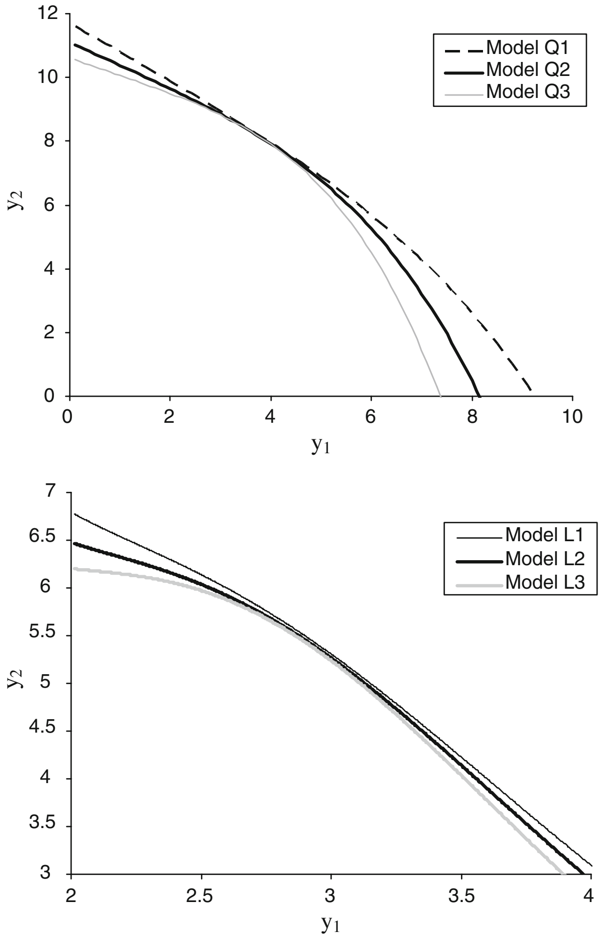

Fig. 1 I. True frontiers of the output set; polynomial technologies. II. True frontiers of the output set; translog technologies

The translog technology is given by

$$P^L(x) = \{(y_1, y_2) : \ln(y_2) = \beta_0^L + \beta_1^L \ln(y_1) + \beta_2^L[\ln(y_1)]^2 + \beta_3^L[\ln(y_1)]^3 + \beta_4^L[\ln(y_1)]^4 + x_1^{0.9}x_2^{0.8} \equiv f^L(y_1, x)\}, \; x \in \Re_+^2. \tag{13}$$

We choose the parameter vector $\beta^L = (\beta_0^L, \beta_1^L, \beta_2^L, \beta_3^L, \beta_4^L)$ in the following way:

	Model L1	Model L2	Model L3
β_0^L	3.000	2.845	2.690
β_1^L	−3.500	−3.400	−3.300
β_2^L	3.900	4.000	4.100
β_3^L	−1.500	−1.475	−1.415
β_4^L	−0.140	−0.220	−0.330

As shown in Panel II of Fig. 1, all of the true frontiers produced by these parameters satisfy the concavity property. Model L1 is the "least concave" of the three, whereas Model L3 is associated with the highest rate of increase in the opportunity cost of one output in terms of the other.[8]

Samples of size $N = 50$, 100, 500, and 1,000 observations are drawn from a uniform distribution as $\ln(y_1) \sim \text{Uniform}(0.7, 1.4)$. This particular choice of the support for the logarithm of y_1 guarantees that all of our true output set frontiers have non-decreasing marginal rate of transformation at each value of y_1 in the sample. Finally, the input quantities are generated in the same way as before and the technical inefficiency is introduced in the following way:

$$\ln(y_2) = \ln(\exp\{f^L(y_1)\} - \upsilon). \tag{14}$$

The translog and the quadratic output distance functions are given respectively by

$$\begin{aligned} D(x, y) = \exp\{&\gamma_0 + \gamma_1 \ln(y_1) + \gamma_2 \ln(y_2) + \frac{\gamma_{11}}{2}(\ln(y_1))^2 \\ &+ \frac{\gamma_{22}}{2}(\ln(y_2))^2 + \gamma_{12} \ln(y_1)\ln(y_2) \\ &+ \gamma_3 \ln(x_1) + \gamma_4 \ln(x_2) + \frac{\gamma_{33}}{2}(\ln(x_1))^2 \\ &+ \frac{\gamma_{44}}{2}(\ln(x_2))^2 + \gamma_{34} \ln(x_1)\ln(x_2) \\ &+ \gamma_{13} \ln(y_1)\ln(x_1) + \gamma_{14} \ln(y_1)\ln(x_2) \\ &+ \gamma_{23} \ln(y_2)\ln(x_1) + \gamma_{24} \ln(y_2)\ln(x_2)\}, \end{aligned} \tag{15}$$

$$\begin{aligned} \vec{D}(x, y) = {}&\delta_0 + \delta_1 y_1 + \delta_2 y_2 + \frac{\delta_{11}}{2} y_1^2 + \frac{\delta_{22}}{2} y_2^2 + \delta_{12} y_1 y_2 \\ &+ \delta_3 x_1 + \delta_4 x_2 + \frac{\delta_{33}}{2} x_1^2 + \frac{\delta_{44}}{2} x_2^2 + \delta_{34} x_1 x_2 \\ &+ \delta_{13} y_1 x_1 + \delta_{14} y_1 x_2 + \delta_{23} y_2 x_1 + \delta_{24} y_2 x_2. \end{aligned} \tag{16}$$

The parameters of these functions are computed using the linear programming procedure of Aigner and Chu (1968).[9] We consider 200 replications for each of the nine model types and both parameterizations.[10]

[8] As with the polynomial technologies, a number of alternative specifications for true translog models were considered at the initial stages of the study. Since the quality of approximation provided by the quadratic directional distance function versus the translog Shephard distance function was very similar across various true technologies, we decided to summarize the results from just three of them.

[9] Linear programming methodology facilitates a straightforward modeling of all of the functions' properties, such as representation, monotonicity, as well as homogeneity (in the case of the Shephard distance function), and translation (in the case of the directional distance function). See Färe et al. (2005) or Vardanyan and Noh (2006) for a more detailed analysis of the computation procedures.

[10] We have also considered models that include interaction terms between y_1 and the inputs in the expressions representing true technology. Since all simulation results remained qualitatively unaffected, we imposed separability between inputs and outputs in our data generating process (DGP). We note that such assumption is not incorporated in equations (15) and (16), reflecting the potential ignorance about the true DGP and our desire to evaluate the performance of generalized quadratic specifications.

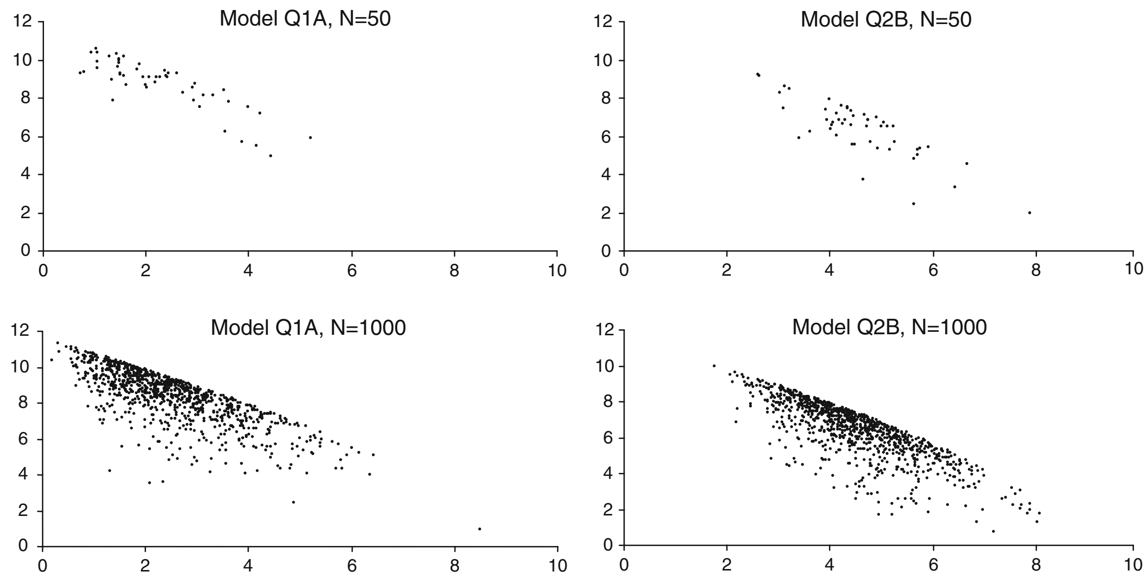

Fig. 2 Scatter plots of type-A and type-B models

The empirical analogues of the frontier of the production technology can be recovered using the vectors of parameter estimates $\hat{\gamma}$ and $\hat{\delta}$ and by assuming technical efficiency for every observation in the sample, i.e., $\vec{D}^n(x, y; g) = 0$ and $D^n(x, y) = 1$ for all $n = 1,\ldots,N$. Using the first output y_{1n} and assuming $x_{kn} = \bar{x}_k,\ k = 1, 2$, we can solve N quadratic equations and simulate the optimal output quantities $y^*_{2n}(\hat{\gamma})$ and $y^*_{2n}(\hat{\delta})$, which put every observation on the frontier of the estimated set $\hat{P}(\bar{x})$, thereby allowing us to compare it to the true set $P(\bar{x})$and assess the quality of approximation provided by the translog versus the quadratic parameterizations.[11]

We choose the following three benchmarks in order to investigate the desirability of the parametric approximations: (1) the average Euclidean distance between the true and simulated quantity of the second output; (2) the average relative shadow price discrepancy; and (3) the mean Euclidean distance between the true and estimated Morishima elasticities of substitution for the two parameterizations.

Our first benchmark is obtained using the true and simulated quantities of the second output. It is defined as

$$\bar{\Delta}(\cdot) = N^{-1}\sqrt{\sum\nolimits_{n=1}^{N} \left[y^*_{2n}(\cdot) - (y_{2n} + \upsilon_n)\right]^2}. \tag{17}$$

The second benchmark can be interpreted as the average discrepancy between the true and estimated marginal rate of transformation evaluated at frontier points. From duality theory, the relative shadow price can be defined as (Färe and Primont 1995; Färe and Grosskopf 2004)

$$\frac{p_1}{p_2} = \frac{\partial D(x, y)/\partial y_1}{\partial D(x, y)/\partial y_2} = \frac{\partial \vec{D}(x, y; g)/\partial y_1}{\partial \vec{D}(x, y; g)/\partial y_2}. \tag{18}$$

Hence, the average Euclidean distance between the true and the estimated relative shadow price is equal to

$$\bar{\Omega}(\hat{\gamma}) = N^{-1}\sum_{n=1}^{N}\sqrt{\left[p_n + \frac{\partial \ln D(x, y_{1n}, y_{2n})/\partial \ln(y_1)}{\partial \ln D(x, y_{1n}, y_{2n})/\partial \ln(y_2)}\frac{y_{2n}}{y_{1n}}\right]^2} \tag{19}$$

and

$$\bar{\Omega}(\hat{\delta}) = N^{-1}\sum_{n=1}^{N}\sqrt{\left[p_n + \frac{\partial \vec{D}(x, y_{1n}, y_{2n}; g)/\partial y_1}{\partial \vec{D}(x, y_{1n}, y_{2n}; g)/\partial y_2}\right]^2}, \tag{20}$$

where p_n is the negative of the true shadow price for observation n. Note that $p_n = \frac{\partial f^Q(y_{1n}, x)}{\partial y_1}$ in the case of polynomial technologies and $p_n = \frac{\partial f^L(y_{1n}, x)}{\partial \ln(y_1)}\frac{y_{2n}}{y_{1n}}$ for translog technologies.

Finally, our third benchmark assesses the relative error in the approximation of the Morishima elasticity of substitution, a measure of the curvature. It is defined as $\partial \ln(p_1/p_2)/\partial \ln(y_2/y_1)$ and we have

$$e(\hat{\gamma}) = 1 - \frac{\partial^2 \ln D(x, y)\big/\partial(\ln y_1)^2}{\partial \ln D(x, y)/\partial \ln(y_1)} + \frac{\partial^2 \ln D(x, y)/\partial \ln(y_1)\partial \ln(y_2)}{\partial \ln D(x, y)/\partial \ln(y_2)}, \tag{21}$$

[11] One can alternatively use y_2 to simulate the optimal quantity of the first output.

and

$$e(\hat{\delta}) = y_1 \left(\frac{\partial^2 \vec{D}(x,y,g) \big/ \partial y_1 \partial y_2}{\partial \vec{D}(x,y,g) \big/ \partial y_2} - \frac{\partial^2 \vec{D}(x,y,g) \big/ \partial y_1^2}{\partial \vec{D}(x,y,g) \big/ \partial y_1} \right). \tag{22}$$

Therefore, the average Euclidean distance between the true and estimated elasticity is equal to

$$\bar{\mathrm{E}}(\hat{\gamma}) = N^{-1} \sum\nolimits_{n=1}^{N} \sqrt{\left[e_n + 1 - \frac{\partial^2 \ln D(x, y_{1n}, y_{2n}) \big/ \partial (\ln y_1)^2}{\partial \ln D(x, y_{1n}, y_{2n}) / \partial \ln(y_1)} + \frac{\partial^2 \ln D(x, y_{1n}, y_{2n}) / \partial \ln(y_1) \partial \ln(y_2)}{\partial \ln D(x, y_{1n}, y_{2n}) / \partial \ln(y_2)} \right]^2}, \tag{23}$$

and

$$\bar{\mathrm{E}}(\hat{\gamma}) = N^{-1} \sum\nolimits_{n=1}^{N} \sqrt{\left[e_n + 1 - \frac{\partial^2 \ln D(x, y_{1n}, y_{2n}) \big/ \partial (\ln y_1)^2}{\partial \ln D(x, y_{1n}, y_{2n}) / \partial \ln(y_1)} + \frac{\partial^2 \ln D(x, y_{1n}, y_{2n}) / \partial \ln(y_1) \partial \ln(y_2)}{\partial \ln D(x, y_{1n}, y_{2n}) / \partial \ln(y_2)} \right]^2}, \tag{23}$$

where e_n is the negative of the true elasticity of substitution for observation n. Also, note that $e_n = \frac{\partial^2 f^L(y_{1n},x) \big/ \partial \ln(y_1)^2}{\partial f^L(y_{1n},x) / \partial \ln(y_1)} - 1$ and $e_n = y_{1n} \frac{\partial^2 f^Q(y_{1n},x) \big/ \partial y_1^2}{\partial f^Q(y_{1n},x) / \partial y_1}$ for translog and polynomial technologies, respectively.

3 Results

The results of our experiment are summarized in Table 1 for the true polynomial technologies and Table 2 for the true translog technologies. For each of the quadratic models we specify three directional vectors $g \equiv (g_{y_1}, g_{y_2})$, namely $g = (1{,}5)$, $g = (1{,}1)$, and $g = (5{,}1)$.[12] We assess both the translog and the quadratic estimates using the three benchmarks and their weighted average. The first benchmark was assigned a 50 percent weight, whereas the shadow price and the elasticity discrepancy were assigned a 25 percent weight each. We also report the fraction of models that resulted in a normal convergence, denoted by "%" in the second column of the tables, since in some of our models this fraction declined quite considerably with an increase in sample size.

Figures 3 and 4, which contain the plots corresponding to just two polynomial production technologies Model Q1A and Model Q3B for sample sizes $N = 50$ and $N = 1{,}000$, suggest that the quadratic directional distance function performs better than the translog Shephard distance function. For example, all of the quadratic frontier estimates (panels A through C) seem to be doing a better job of enveloping the true frontier compared to the translog estimate (panel D). This result is not surprising, since quadratic functions are simply second-order approximations to these true technologies, which are fourth-order polynomials. In fact, the Shephard distance functions produce frontier estimates that have a wrong curvature in *all* of the true polynomial models. We attempted to impose concavity by restricting the estimated elasticity of substitution to be non-positive, but none of these restricted specifications managed to converge to a normal solution *in type-A models*. As a result, we only report their convex plots in panel D of Fig. 3, but not their benchmark values. Interestingly, the same concavity restrictions imposed on the parameters of the Shephard distance function produce the desired concave contour of the frontier *in type-B models*, as is illustrated in panel D of Fig. 4, provided the sample size is not very large.[13] However, even when the estimated translog frontier is concave, it does not seem to fare as well as quadratic estimates (panels A through C of Fig. 4). Note that due to this need for curvature constraints the fraction of converging translog parameterizations

[12] Note that the methods we use to add technical inefficiency in (12) and (14) ignore different scale effects entailed by these directional vectors. For a detailed discussion of scaling using directional vectors see Färe and Grosskopf (2004).

[13] We do not report any of the benchmark values unless the fraction of converging simulations is greater than one.

Table 1 Results of the Monte Carlo experiment; polynomial technologies

Model Q1A	%[a]	Mean Euclidean output distance $[\bar{\Delta}(\cdot)]$	Mean Euclidean shadow price distance $[\bar{\Omega}(\cdot)]$	Mean Euclidean elasticity of substitution distance $[\bar{\mathrm{E}}(\cdot)]$	Weighted Euclidean distance	Model Q1B	%	Mean Euclidean output distance $[\bar{\Delta}(\cdot)]$	Mean Euclidean shadow price distance $[\bar{\Omega}(\cdot)]$	Mean Euclidean elasticity of substitution distance $[\bar{\mathrm{E}}(\cdot)]$	Weighted Euclidean distance
Quadratic directional output distance function						Quadratic directional output distance function					
$g = (1,5)$						$g = (1,5)$					
$N = 50$	67.8	115.43	119.96	230.63	145.35	$N = 50$	62.2	124.00	116.10	384.34	187.11
$N = 100$	76.0	65.64	62.00	130.82	81.02	$N = 100$	64.4	62.10	63.59	222.04	102.46
$N = 500$	70.1	20.90	25.75	80.89	37.11	$N = 500$	69.1	20.28	29.08	169.97	59.90
$N = 1{,}000$	76.6	14.42	20.39	77.09	31.58	$N = 1{,}000$	73.1	16.10	24.95	170.93	57.02
$g = (1,1)$						$g = (1,1)$					
$N = 50$	59.0	124.17	108.63	116.93	118.47	$N = 50$	65.9	118.19	113.28	239.90	147.39
$N = 100$	61.8	66.70	58.68	89.45	70.38	$N = 100$	66.6	63.44	62.90	217.52	101.83
$N = 500$	64.4	18.85	24.40	78.78	35.22	$N = 500$	69.1	18.12	24.67	249.78	77.67
$N = 1{,}000$	66.6	13.85	20.80	80.15	32.16	$N = 1{,}000$	71.9	13.50	21.88	261.69	77.64
$g = (5,1)$						$g = (5,1)$					
$N = 50$	59.7	118.20	128.76	100.52	116.42	$N = 50$	64.7	120.16	115.15	388.76	186.06
$N = 100$	62.8	72.06	78.78	100.91	80.95	$N = 100$	70.9	69.15	75.30	402.62	154.05
$N = 500$	72.3	26.99	32.22	108.95	48.79	$N = 500$	75.6	24.45	33.48	414.29	124.17
$N = 1{,}000$	75.4	20.38	25.32	111.21	44.32	$N = 1{,}000$	74.4	18.22	28.14	417.63	120.55
Translog Shephard output distance function						Translog Shephard output distance function					
$N = 50$	0.0	–	–	–	–	$N = 50$	55.5	205.28	118.73	463.28	248.14
$N = 100$	0.0	–	–	–	–	$N = 100$	56.5	226.93	96.46	424.63	243.74
$N = 500$	0.0	–	–	–	–	$N = 500$	14.7	428.32	78.77	395.98	332.85
$N = 1{,}000$	0.0	–	–	–	–	$N = 1{,}000$	3.2	453.16	77.86	401.16	346.33

Table 1 continued

Model Q2A	%	Mean Euclidean output distance $[\bar{\Delta}(\cdot)]$	Mean Euclidean shadow price distance $[\bar{\Omega}(\cdot)]$	Mean Euclidean elasticity of substitution distance $[\bar{\mathrm{E}}(\cdot)]$	Weighted Euclidean distance	Model Q2B	%	Mean Euclidean Output Distance $[\bar{\Delta}(\cdot)]$	Mean Euclidean shadow price distance $[\bar{\Omega}(\cdot)]$	Mean Euclidean elasticity of substitution distance $[\bar{\mathrm{E}}(\cdot)]$	Weighted Euclidean distance
Quadratic directional output distance function						Quadratic directional output distance function					
$g = (1,5)$						$g = (1,5)$					
$N = 50$	69.1	119.95	114.02	310.01	165.98	$N = 50$	67.1	118.27	116.40	507.26	215.05
$N = 100$	72.0	67.87	77.41	242.82	113.99	$N = 100$	64.3	66.51	74.27	373.03	145.08
$N = 500$	72.8	29.73	48.87	196.75	75.77	$N = 500$	66.6	27.55	47.86	353.20	114.04
$N = 1{,}000$	77.1	25.57	42.57	193.54	71.81	$N = 1{,}000$	67.1	26.43	47.42	359.33	114.90
$g = (1,1)$						$g = (1,1)$					
$N = 50$	58.3	122.69	108.37	203.00	139.19	$N = 50$	64.6	124.98	108.30	578.88	234.29
$N = 100$	63.4	65.90	63.73	188.08	95.90	$N = 100$	64.3	68.14	69.22	624.91	207.60
$N = 500$	63.1	24.72	38.53	199.72	71.92	$N = 500$	66.6	20.81	39.31	651.18	183.03
$N = 1{,}000$	64.0	22.74	36.54	206.83	72.22	$N = 1{,}000$	71.7	18.44	37.33	654.05	182.06
$g = (5,1)$						$g = (5,1)$					
$N = 50$	49.5	114.78	122.32	295.77	191.91	$N = 50$	59.4	120.94	142.16	976.79	340.21
$N = 100$	60.6	67.49	79.48	292.04	126.63	$N = 100$	62.3	79.88	109.69	985.87	313.83
$N = 500$	65.0	34.44	44.06	300.35	103.32	$N = 500$	68.9	53.29	93.84	996.86	299.32
$N = 1{,}000$	69.1	32.55	40.04	300.62	101.44	$N = 1{,}000$	72.6	58.51	97.61	1,004.03	304.66
Translog Shephard output distance function						Translog Shephard output distance function					
$N = 50$	0.0	–	–	–	–	$N = 50$	48.3	225.67	179.71	850.44	370.37
$N = 100$	0.0	–	–	–	–	$N = 100$	37.5	286.18	174.90	834.37	395.41
$N = 500$	0.0	–	–	–	–	$N = 500$	2.2	500.56	202.65	930.06	533.46
$N = 1{,}000$	0.0	–	–	–	–	$N = 1{,}000$	<1.0	–	–	–	–

Table 1 continued

Model Q3A	%	Mean Euclidean output distance $[\bar{\Delta}(\cdot)]$	Mean Euclidean shadow price distance $[\bar{\Omega}(\cdot)]$	Mean Euclidean elasticity of substitution distance $[\bar{\mathrm{E}}(\cdot)]$	Weighted Euclidean distance	Model Q3B	%	Mean Euclidean output distance $[\bar{\Delta}(\cdot)]$	Mean Euclidean shadow price distance $[\bar{\Omega}(\cdot)]$	Mean Euclidean elasticity of substitution distance $[\bar{\mathrm{E}}(\cdot)]$	Weighted Euclidean distance
Quadratic directional output distance function						Quadratic directional output distance function					
$g = (1,5)$						$g = (1,5)$					
$N = 50$	71.3	122.27	130.72	667.47	260.69	$N = 50$	64.8	118.93	115.78	625.41	244.77
$N = 100$	72.0	71.45	96.81	439.85	169.89	$N = 100$	62.3	66.45	75.19	531.40	184.87
$N = 500$	72.3	44.21	75.08	367.10	132.65	$N = 500$	65.5	29.61	53.99	525.71	159.73
$N = 1,000$	71.0	44.07	73.25	361.88	130.82	$N = 1,000$	63.1	30.96	56.53	539.90	164.59
$g = (1,1)$						$g = (1,1)$					
$N = 50$	63.3	121.06	110.00	358.69	177.70	$N = 50$	61.5	129.21	156.97	1,113.20	382.15
$N = 100$	59.5	64.65	70.94	348.88	137.28	$N = 100$	68.8	78.98	128.87	1,132.37	354.80
$N = 500$	62.0	28.21	48.61	361.10	116.53	$N = 500$	71.0	62.88	124.67	1,193.58	361.00
$N = 1,000$	61.3	30.82	48.36	368.06	119.52	$N = 1,000$	68.0	69.63	128.33	1,212.97	370.14
$g = (5,1)$						$g = (5,1)$					
$N = 50$	40.2	138.58	175.17	579.41	257.94	$N = 50$	53.3	156.58	245.51	1,599.83	539.62
$N = 100$	44.3	105.19	153.96	597.12	240.37	$N = 100$	52.5	119.50	227.52	1,610.27	519.20
$N = 500$	49.8	112.46	165.04	606.24	249.05	$N = 500$	60.3	126.44	228.31	1,629.83	527.76
$N = 1,000$	55.8	122.32	160.23	609.44	253.58	$N = 1,000$	64.8	138.23	232.12	1,633.65	535.56
Translog Shephard output distance function						Translog Shephard output distance function					
$N = 50$	0.0	–	–	–	–	$N = 50$	35.7	314.91	317.94	1,354.54	575.58
$N = 100$	0.0	–	–	–	–	$N = 100$	19.0	390.29	331.72	1,409.77	630.52
$N = 500$	0.0	–	–	–	–	$N = 500$	<1.0	–	–	–	–
$N = 1,000$	0.0	–	–	–	–	$N = 1,000$	<1.0	–	–	–	–

Each of the mean benchmarks has been multiplied by 10^3

[a] Shows the percent of simulations that converged to a normal solution

Table 2 Results of the Monte Carlo experiment; translog technologies

Model L1	%[a]	Mean Euclidean output distance $[\bar{\Delta}(\cdot)]$	Mean Euclidean shadow price distance $[\bar{\Omega}(\cdot)]$	Mean Euclidean elasticity of substitution distance $[\bar{\mathrm{E}}(\cdot)]$	Weighted Euclidean distance	Model L2	%	Mean Euclidean output distance $[\bar{\Delta}(\cdot)]$	Mean Euclidean shadow price distance $[\bar{\Omega}(\cdot)]$	Mean Euclidean elasticity of substitution distance $[\bar{\mathrm{E}}(\cdot)]$	Weighted Euclidean distance
Quadratic directional output distance function						Quadratic directional output distance function					
$g = (1,5)$						$g = (1,5)$					
$N = 50$	74.3	1,056.24	864.82	2,087.17	1,266.12	$N = 50$	77.0	1,099.20	1,574.73	2,777.10	1,637.56
$N = 100$	72.2	941.17	723.92	2,023.46	1,157.43	$N = 100$	75.8	889.87	1,646.19	2,694.08	1,530.00
$N = 500$	25.3	907.75	707.20	2,053.22	1,143.98	$N = 500$	25.3	859.38	1,699.24	2,745.74	1,540.94
$N = 1{,}000$	23.8	943.38	670.56	2,049.94	1,151.81	$N = 1{,}000$	25.0	909.47	1,706.38	2,753.10	1,569.60
$g = (1,1)$						$g = (1,1)$					
$N = 50$	76.2	921.71	845.38	2,069.57	1,189.59	$N = 50$	76.5	894.21	1,808.82	2,812.67	1,602.48
$N = 100$	81.2	813.48	824.05	2,084.61	1,133.91	$N = 100$	84.0	788.26	1,790.19	2,845.71	1,553.10
$N = 500$	32.9	953.41	756.79	2,075.40	1,184.75	$N = 500$	35.0	852.99	1,747.63	2,832.69	1,571.58
$N = 1{,}000$	33.8	1,080.13	733.32	2,079.90	1,243.37	$N = 1{,}000$	31.5	963.74	1,738.54	2,829.44	1,623.86
$g = (5,1)$						$g = (5,1)$					
$N = 50$	5.8	1,202.13	1,003.26	2,106.68	1,378.55	$N = 50$	5.3	1,106.58	1,870.25	2,852.13	1,733.89
$N = 100$	3.6	852.24	1,006.06	2,098.88	1,202.36	$N = 100$	2.8	953.29	1,888.92	2,828.35	1,655.96
$N = 500$	7.8	753.25	876.87	2,073.11	1,114.12	$N = 500$	7.9	779.47	1,830.50	2,827.23	1,554.17
$N = 1{,}000$	8.3	795.14	857.41	2,076.28	1,130.99	$N = 1{,}000$	8.4	856.68	1,827.84	2,828.66	1,592.46
Translog Shephard output distance function						Translog Shephard output distance function					
$N = 50$	58.8	1,155.19	1,211.56	1,977.01	1,374.73	$N = 50$	54.8	1,135.67	1,350.01	2,717.65	1,584.75
$N = 100$	34.3	980.44	1,132.26	2,024.72	1,279.46	$N = 100$	32.7	1,007.25	1,202.48	2,740.91	1,489.47
$N = 500$	2.4	1,900.91	793.80	2,054.47	1,662.52	$N = 500$	1.8	1,651.56	949.55	2,764.41	1,754.27
$N = 1{,}000$	<1.0	–	–	–	–	$N = 1{,}000$	<1.0	–	–	–	–

Table 2 continued

Model L3	%	Mean Euclidean output distance $[\bar{\Delta}(\cdot)]$	Mean Euclidean shadow price distance $[\bar{\Omega}(\cdot)]$	Mean Euclidean elasticity of substitution distance $[\bar{\mathrm{E}}(\cdot)]$	Weighted Euclidean distance
Quadratic directional output distance function					
$g = (1,5)$					
$N = 50$	75.5	1,167.22	1,208.60	4,119.96	1,915.75
$N = 100$	73.8	1,058.69	1,115.60	4,152.17	1,848.29
$N = 500$	24.8	932.81	1,160.60	4,330.91	1,839.28
$N = 1,000$	23.1	993.97	1,244.68	4,358.44	1,897.77
$g = (1,1)$					
$N = 50$	77.8	921.27	1,403.15	4,446.21	1,922.98
$N = 100$	83.8	775.00	1,362.59	4,495.09	1,851.92
$N = 500$	35.5	873.96	1,269.01	4,480.62	1,874.39
$N = 1,000$	38.9	952.36	1,244.23	4,481.40	1,907.59
$g = (5,1)$					
$N = 50$	5.2	1,102.76	1,546.12	4,550.18	2,075.45
$N = 100$	4.0	930.35	1,479.09	4,483.33	1,955.78
$N = 500$	8.3	898.17	1,396.87	4,491.95	1,921.29
$N = 1,000$	9.9	956.33	1,368.01	4,484.08	1,941.19
Translog Shephard output distance function					
$N = 50$	41.6	1,197.41	1,100.68	4,273.83	1,942.33
$N = 100$	16.3	1,137.27	1,140.86	4,344.02	1,939.86
$N = 500$	<1.0	–	–	–	–
$N = 1,000$	<1.0	–	–	–	–

Each of the mean benchmarks has been multiplied by 10^3

[a] Shows the percent of simulations that converged to a normal solution

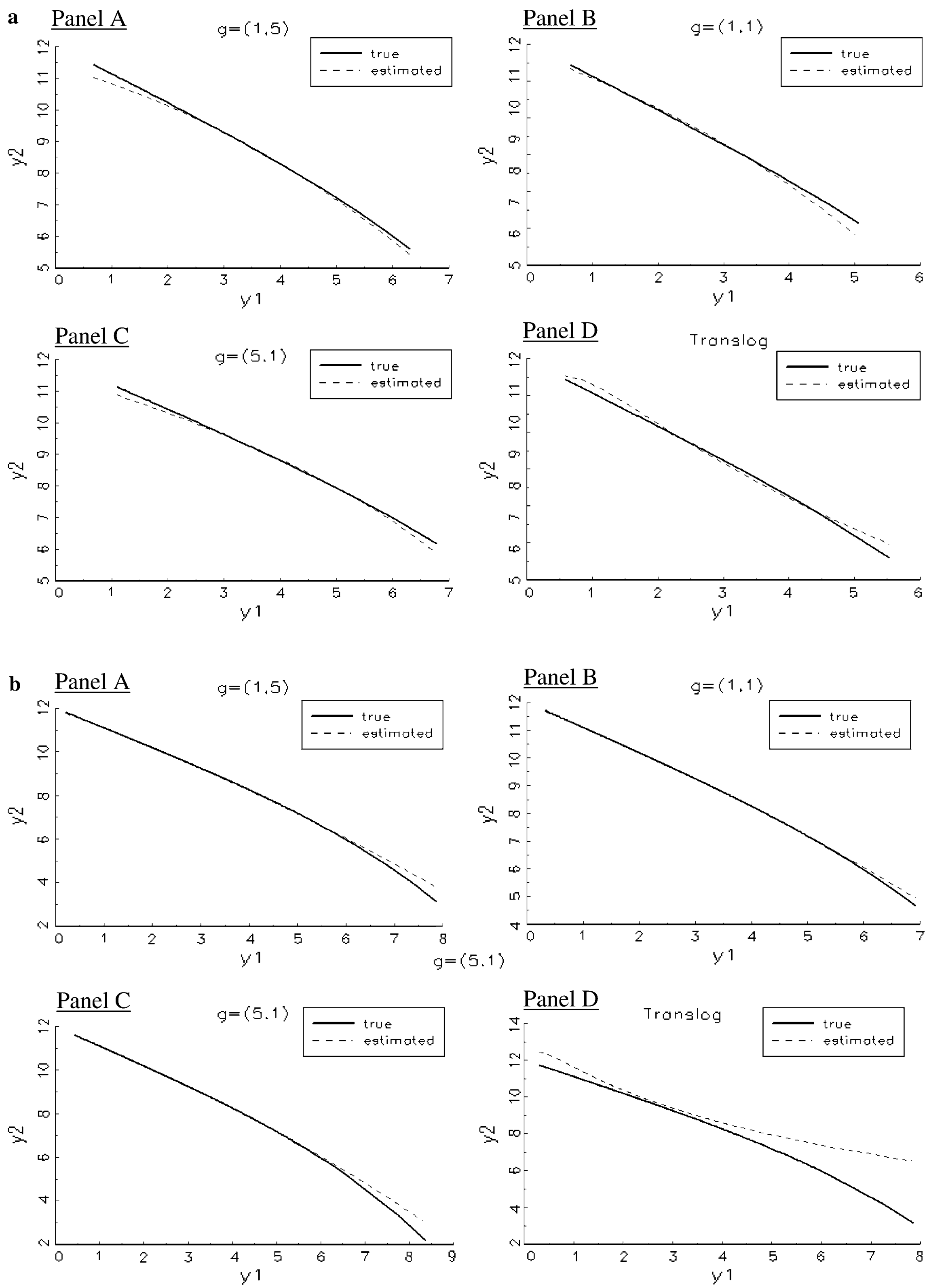

Fig. 3 **a** The true and estimated frontiers of the output set; model Q1A, $N = 50$; **b** model Q1A, $N = 1{,}000$

declines quite dramatically with an increase in the sample size and, for example, is equal to only around 3 percent in Model Q1B when $N = 1{,}000$.

As far as the true translog technologies are concerned, *both* the directional and the Shephard distance function models require that the curvature conditions be imposed.

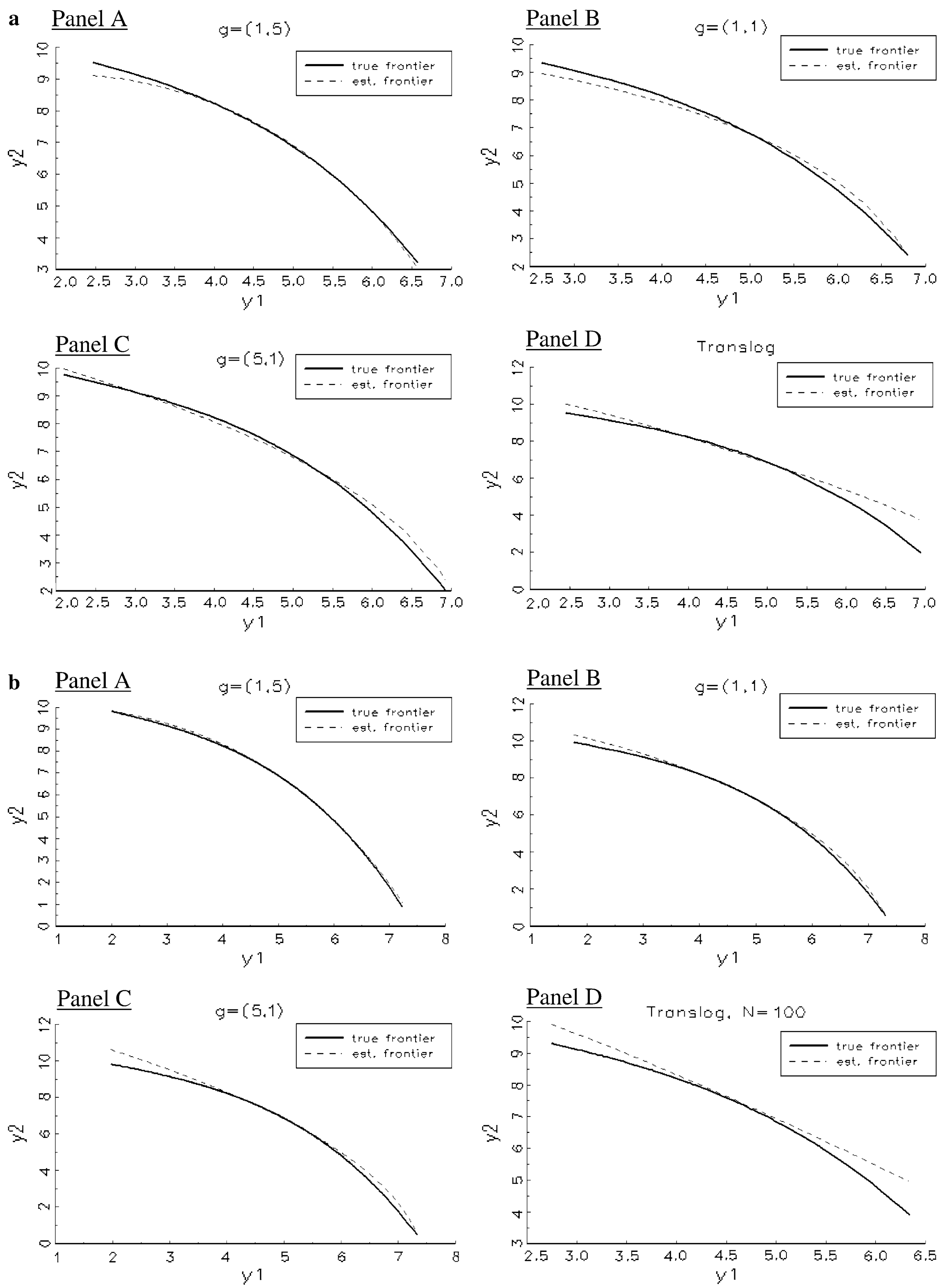

Fig. 4 **a** The true and estimated frontiers of the output set; model Q3B, $N = 50$; **b** model Q3B, $N = 1{,}000$

The plots of these estimated frontiers, which are presented in Fig. 5 only for Model L2, suggest that the difference in the quality of approximation between the two parameterizations is not as significant as in the case of polynomial technologies. However, as shown in the second and the eighth column of Table 2, the fraction of

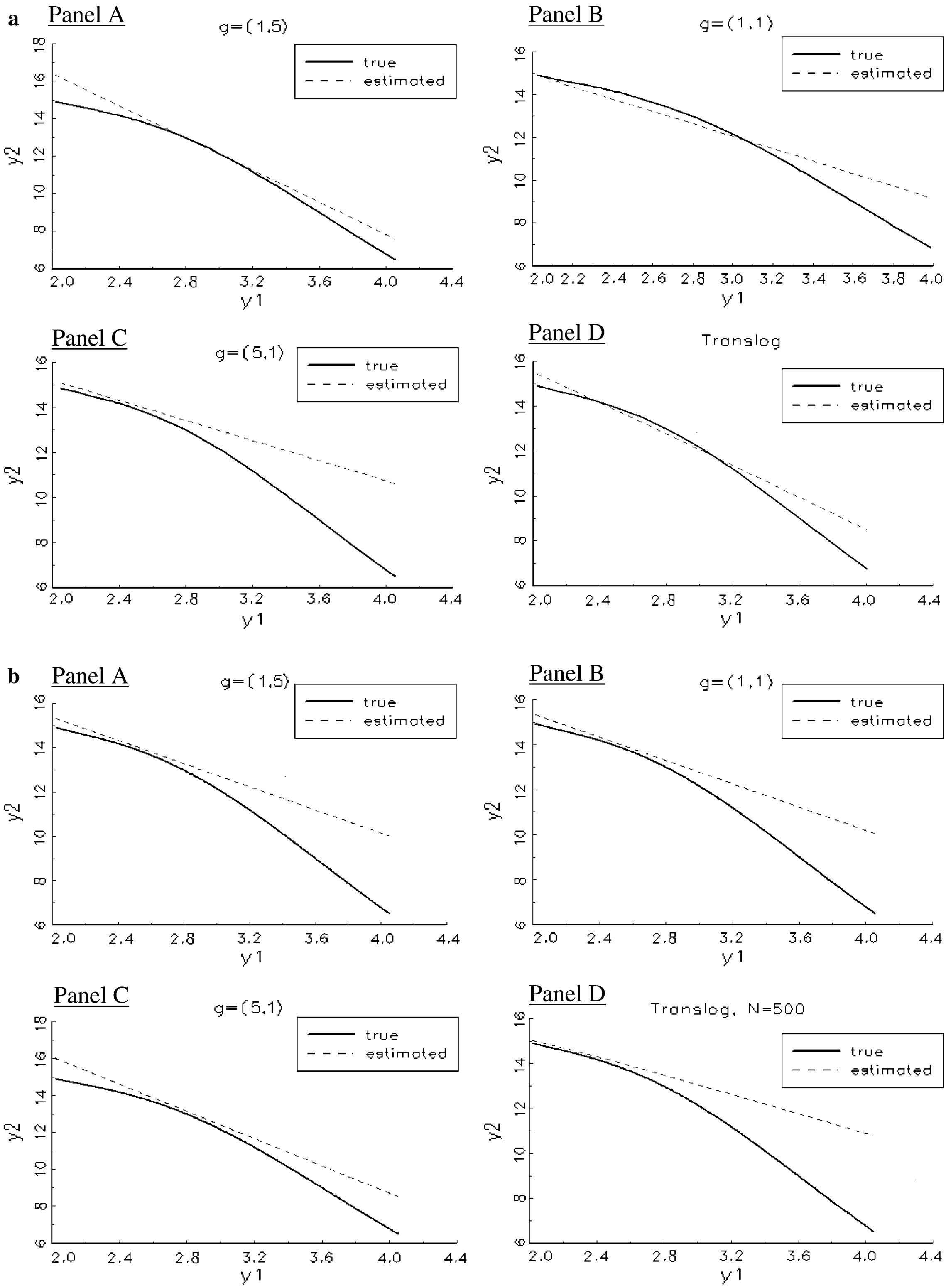

Fig. 5 **a** The true and estimated frontiers of the output set; model L2, $N = 50$; **b** model L2, $N = 1{,}000$

converging specifications drops sharply as the sample size is increased, and the rate of this decline is more pronounced than in polynomial models. To some extent, this precipitous decline can be explained by a dramatic increase in the number of representation, monotonicity, and especially curvature constraints that must be satisfied in large samples.

A more rigorous assessment of the approximation properties of the quadratic versus translog functions can be performed by comparing their corresponding benchmark values, which we did in Table 1 for polynomial technologies and Table 2 for translog technologies. Recall that three of them were defined in the previous section: the mean Euclidean distance between the true and the simulated quantity of y_2, denoted by $\bar{\Delta}(\cdot)$ and reported in the third and the ninth column of Tables 1 and 2, as well as the average relative shadow price discrepancy $[\bar{\Omega}(\cdot)]$ and the mean Euclidean distance between the true and estimated elasticity of substitution $[\bar{\mathrm{E}}(\cdot)]$.

The numbers suggest that in the case of the true polynomial technologies, the quadratic function's global behavior is clearly superior to that of the translog function.[14] For example, in type-B models the directional distance function's weighted average benchmarks, which steadily decrease as the sample size grows, are always lower than their Shephard distance function counterparts (column 12 of Table 1).[15] On the contrary, with an increase in N the translog's performance is in fact getting worse, not better. For instance, in Model Q2B its average benchmark goes up from 370.4 to 395.4 as the sample size grows from 50 to 100 observations and then continues to rise to 533.5 as the sample is increased to $N = 500$.

As far as the true tranlsog technologies are concerned, the difference in the quality of approximation between the two parameterizations is not always so apparent—a result that is consistent with plots in Fig. 5. For example, for all three directional vectors the quadratic function continues to fare better than the translog in Model L1, but not in Model L2. For example, when $N = 100$ the mean benchmark corresponding to the translog Shephard distance function equals approximately 1,279 and 1,489 in Model L1 and Model L2, respectively (bottom of column 6 and column 12 of Table 2), compared to the maximum average benchmark of 1,202 in Model L1 and 1,656 in Model L2 in the case of the quadratic directional distance function.[16] In other words, the quadratic function can sometimes approximate the true technology better than can the translog function even when the true technology is translog itself!

The behavior of the directional distance functions seems to be affected to some extent by the choice of the directional vector g. For example, the approximation quality achieved in parameterizations that assume $g = (5,1)$ is consistently poorer compared to models based on the other two directional vectors. This difference is most noticeable in type-B models, but is generally never very large. For instance, the mean benchmark associated with this directional vector in Model Q1B when $N = 500$ equals 124.17 (Column 12 of Table 1), compared to the average benchmark values of 59.9 and 77.7 for the other two directional vectors from the same model. In addition, only a relatively small fraction of quadratic parameterizations that are based on $g = (5,1)$ managed to converge to a normal solution in all three of our true translog models (column 2 and column 8 of Table 2). Hence, the estimation results are not entirely invariant to the choice of the directional vector.

The fraction of translog simulations converging to a normal solution appears to be susceptible to the extent of the curvature of the underlying true frontier. For example, in the case of polynomial technologies the fraction of converging translog models falls from more than 56% in Model Q1B to just 19% in Model Q3B when $N = 100$ (bottom of column 8 of Table 1). Similarly, in the case of the true translog technologies this percentage drops from about 34% in Model L1 to 16% in Model L3 (bottom of column 2 of Table 2).

Finally, among the three types of polynomial technologies that we consider the best quality of approximation is achieved in Model Q1, which is associated with the lowest average marginal rate of transformation. After that, the performance gradually deteriorates in both the translog and quadratic parameterizations, as is reflected by a steady increase in the values of all mean benchmarks reported in column 6 for type-A models and column 12 for type-B models. One possible explanation to this trend may be our assumption regarding the true production technologies, which are polynomials of order four, whereas the functions that are used to approximate them are processes of order two. Consequently, these second-order parameterizations would be better suited to approximate the frontiers that are relatively "flat," such as in Model Q1 or Model Q2. As far as the true translog technologies are concerned, the weighted average benchmarks reach their lowest level in Model L1 (bottom of column 6 of Table 2) and steadily increase thereafter.

To summarize, the unrestricted translog function tends to produce convex estimates of the output set frontier and is rather inflexible if subjected to concavity restrictions. These restrictions are always necessary, and the convergence can be achieved only if the sample size is not very

[14] Note that the benchmark values associated with the Shephard distance function are not reported for any of our type-A models due to the wrong (convex) curvature of the corresponding frontier estimates and the failure of the estimation algorithms to converge after the imposition of appropriate curvature constraints.

[15] There are a few instances in quadratic models when the average benchmark value increased as the sample size went up from 100 to 500 to 1,000 observations. The rate of this increase was the highest in Model Q3A and Model Q3B, i.e. the two frontiers with the most curvature.

[16] Note that these maxima are obtained across three quadratic parameterizations, each of which corresponds to a specific value of the directional vector, and that both of the maxima reported are from parameterizations that assume $g = (5,1)$.

large. Quite often the tranlsog distance function fails to converge to a normal solution altogether. In addition, the behavior of translog parameterizations deteriorates significantly in models whose output set frontier has plenty of curvature. Studies have shown that the translog functional form performs well when used to approximate cost functions (Guilkey et al. 1983), and is a popular choice when parameterizing Shephard *input* distance functions (Atkinson et al. 2003a, b). However, both the cost function and the input distance functions are defined with respect to the input sets $L(y) = \{x : (x, y) \in T\}$, which have convex boundaries, making translog a natural choice for parameterizing the technology.

On the other hand, depending on the nature of true technology, the unrestricted quadratic function can produce either convex or concave estimates of the output set frontier. Unlike with the translog function, concavity can always be established simply by imposing appropriate restrictions. Hence, although it is certainly not perfect, the quadratic function seems to be much more flexible compared to translog for the family of true technologies included in our study. It performs well regardless of the type of the true technology, as well as in both small and big samples.

Our analysis can be extended by expanding the class of functional forms to include nonlinear-in-parameters specifications, as well as by assigning a parametric structure to the component that represents technical inefficiency. Future research can be expanded even further by generalizing the true technologies to include allocative inefficiency terms as well.

4 Conclusion

The modeling of production technologies with output distance functions can be performed using either a translog Shephard output distance function or a quadratic directional output distance function. In this paper we have compared the performance of these parameterizations by means of a Monte Carlo experiment that assumes two types of true production technologies—polynomial and translog. Our results are in line with earlier studies by Vardanyan and Noh (2006) and Färe et al. (2008) and indicate that quadratic models outperform translog parameterizations regardless of the type of the true technology. We also demonstrate that the translog models are characterized by rather poor economic approximation properties and are quite inflexible when subjected to curvature restrictions.

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