Math Camp - Professor Martins
Homework 2
August 2026

1. Let $f \in B(\mathbb{X}, \mathbb{Y})$. Show that $(B(\mathbb{X}, \mathbb{Y}), \|f\|)$ is a normed vector space.
2. Show that if $A \in \mathcal{M}^{n \times n}$ is symmetric, the eigenvalues of A are real numbers. Also, show that if two eigenvalues are distinct the corresponding eigenvectors are linearly independent.
3. Show that:
   - Items 1, 2 on p. 66 of your notes are true
   - X, the right-inverse of A on p. 68 of your notes is nonsingular
   - If a matrix B is obtained by interchanging any two rows of A, $det(B) = -det(A)$
   - If a matrix A has two rows that are equal, $det(A) = 0$
   - If the rows of A are linearly dependent $det(A) = 0$
   - $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
4. Suppose $A \in \mathcal{M}^{n \times n}$ is symmetric with positive eigenvalues. Show that a nonsingular matrix B can be found such that $A = BB^T$. Also, show that there exists a matrix C such that $A = CC$.
5. Verify the formula for partitioned inverse on item 2 of page 72 of your notes.
6. Consider a vector space whose elements are $m \times n$ matrices with components that are elements of $\mathbb{R}$. In this space, let the $m \times n$ matrix of zeros be the null vector, the sum $s = A + B$ be defined element-wise, and the scalar product $P = aA$ where $a \in \mathbb{R}$ be defined by multiplying each element of A by a. Now, consider the function,
$$f(A) = \sup_{\|x\|_E \neq 0} \frac{\|Ax\|_E}{\|x\|_E}$$
where $\|x\|_E = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$ is the Euclidean norm for $x \in \mathbb{R}^n$. Is f a norm for the space of $m \times n$ matrices? Prove. Show that $\|Ax\|_E \leq f(A)\|x\|_E$.
7. Let $\mathbb{X}$ be a normed vector space. The space of all bounded (continuous) linear functionals on $\mathbb{X}$ is called the normed dual of $\mathbb{X}$ and is denoted by $\mathbb{X}^*$. Show that $\mathbb{X}^*$ is a Banach space.
8. Prove Theorem 66 in your notes.