Math Camp - Professor Martins
Homework 3
August 2026

1. Show that $(\mathbb{R}^n, \|\cdot\|_E)$, where $\|\cdot\|_E$ is the Euclidean norm, is a Banach space.
2. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X, \|\cdot\|_X)$. Show that if some subsequence $\{x_{n(k)}\}$ converges to $x^* \in X$, then $x_n \to x^*$.
3. Cantor's Intersection Theorem (Theorem 71) requires that each $Q_k$ be closed and that $Q_1$ be bounded. Show by counterexample that neither hypothesis can be dropped.
4. Let $A = (0, 1] \subset \mathbb{R}$ with the absolute-value norm. Show directly that $A$ is not complete, and reconcile this with Theorem 74.
5. Complete the proof of Theorem 79.
6. Let $f : (0, 1] \to \mathbb{R}$, $f(x) = x^{-1}$. Show $f$ is continuous but unbounded, and explain why this does not contradict Theorem 81. Then show $f$ is not uniformly continuous on $(0, 1]$, and explain why this does not contradict Heine's Theorem (Theorem 83).
7. Let $g : (0, 1] \to \mathbb{R}$, $g(x) = x^2$. Show that $g$ *is* uniformly continuous on $(0, 1]$. What does this say about the logical status of compactness in Theorem 83?
8. (Existence of a consumer optimum.) Let $p \in \mathbb{R}^n$ with $p_i > 0$ for all $i$, let $w > 0$, and define the budget set
$$B(p, w) = \{x \in \mathbb{R}^n : x_i \geq 0 \ \forall i,\ p^T x \leq w\}.$$
Show $B(p, w)$ is compact and nonempty, and conclude that any continuous $u : \mathbb{R}^n_+ \to \mathbb{R}$ attains a maximum on $B(p, w)$. Show by example that the conclusion can fail if $p_i = 0$ for some $i$.
9. Prove that the derivatives of a sum, a polynomial and a ratio in item 3 of Remark 23 have the stated forms.
10. Let $f : (a, b) \to \mathbb{R}$ be of class $C^2$ and let $x_0 \in (a, b)$ satisfy $f^{(1)}(x_0) = 0$ and $f^{(2)}(x_0) < 0$. Using Theorem 90 with $n = 2$, prove that $x_0$ is a strict local maximum of $f$.
11. Extend this to $\mathbb{R}^n$: let $X \subseteq \mathbb{R}^n$ be open, $f \in C^2(X)$, and $x_0 \in X$ with $Df(x_0) = \theta^\mathsf{T}$ and $D^2 f(x_0)$ negative definite. Using Theorem 103, prove that $x_0$ is a strict local maximum.
12. Prove Euler's Theorem (Theorem 106), left as an exercise in the notes: for $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ with continuous partial derivatives on an open cone $X$, $f$ is homogeneous of degree $k$ if and only if $\sum_{i=1}^n \frac{\partial}{\partial i} f(x)\, x_i = k f(x)$ for all $x \in X$.
13. Let $f : (\mathbb{R}^n, \|\cdot\|_E) \to (\mathbb{R}, |\cdot|)$ be given by $f(x) = \sum_{i=1}^n x_i a_i$ for $a \in \mathbb{R}^n$ with $a_i$ the $i$-th component of $a$. Is $f$ linear? Is it bounded? If so, using the norm in definition 42 of your notes, show that $\|f\| = \|a\|_E$.
14. Suppose you bought Amazon stock at time $t$ at price $P_t$. You sold it at time $t + 1$ at price $P_{t+1}$. Financial economists define
$$R_{t+1} := \frac{P_{t+1} - P_t}{P_t}$$
as the net return from holding the stock from time $t$ to $t+1$. Show that $R_{t+1}$ can be well approximated by $\log \frac{P_{t+1}}{P_t}$. Hint: Use the Mean Value Theorem.
15. Let $f : (\mathbb{R}^n, , \|\cdot\|_E) \to (\mathbb{R}, |\cdot|)$ having continuous partial derivatives in each direction $i = 1, \cdots, n$. Obtain the Gâteaux differential at $x$ with increment $h$.

16. Obtain the partial derivatives and directional derivative (in the direction $u$) of each of the following functions defined on $\mathbb{R}^n$: a) $f(x) = a^T x$ for $a \in \mathbb{R}^n$ a fixed point; b) $f(x) = x^T L(x)$ where $L \in L(\mathbb{R}^n, \mathbb{R}^n)$; c) $f(x) = x^T A x$ where $A$ is a symmetric matrix.

    **Answer.** a) First we obtain a directional derivative.

    $$\lim_{h\to 0}\frac{1}{h}(f(x+h)-f(x)) = \lim_{h\to 0}\frac{1}{h}(a^T(x+hu)-a^Tx)) = \lim_{h\to 0}\frac{1}{h}(ha^Tu) = a^Tu.$$

    The partial derivatives are obtained by taking $u^T = e_i^T$ where $e_i$ is the unit coordinate vector. Hence, $\frac{\partial}{\partial_i}(x) = a^T e_i = a_i$.

    b) if $f(x) = x^T L(x)$

    $$\begin{aligned} f(x+hu)-f(x) &= (x+hu)^T L(x+hu) - x^T L(x) \\ &= (x^T + hu^T)(L(x)+hL(u)) - x^T L(x) \\ &= x^T L(x) + hx^T L(u) + hu^T L(x) + h^2 U^T L(u) - x^T L(x) \\ &= hx^T L(u) + hu^T L(x) + h^2 U^T L(u). \end{aligned}$$

    Hence,

    $$\begin{aligned} \lim_{h\to 0}\frac{1}{h}(f(x+h)-f(x)) &= \lim_{h\to 0}\frac{1}{h}(hx^T L(u) + hu^T L(x) + h^2 U^T L(u)) \\ &= \lim_{h\to 0}(x^T L(u) + u^T L(x) + hU^T L(u)) = x^T L(u) + u^T L(x) \end{aligned}$$

    The partial derivatives are obtained by taking $u^T = e_i^T$ where $e_i$ is the unit coordinate vector. Hence, $\frac{\partial}{\partial_i}(x) = x^T L(e_i) + L_i(x)$.

17. Let $y \in \mathbb{R}^n$, let $X$ be an $n \times k$ matrix, and define $Q : \mathbb{R}^k \to \mathbb{R}$ by $Q(\beta) = (y - X\beta)^{\mathsf{T}}(y - X\beta)$. Using Definition 62 *directly* (not a rule of calculus), find the Fréchet differential of $Q$ at $\beta$ and identify the Fréchet derivative $A(\beta)$. In which space does $A(\beta)$ live, per Remark 27? Derive the first-order condition.

18. Calculate and explain how you can use the fundamental theorem of calculus to obtain the value of the following integrals: a) $\int_{-1}^{1}(ax^2 + bx + c)dx$; b) $\int_1^2 \exp(-2x)dx$.

    **Answer.** All that is needed is to find a primitive for each of the functions being integrated. By the Fundamental Theorem of Calculus, if $F$ is a primitive for $f$, i.e., $F^{(1)} = f$ then

    $$F(b) - F(a) = \int_a^b f(t)dt.$$

    For a) a primitive is $F(x) = a\frac{x^3}{3} + b\frac{x^2}{2} + cx$. Hence,

    $$\int_{-1}^{1}(ax^2+bx+c)dx = a\frac{1^3}{3} + b\frac{1^2}{2} + c1 - (a\frac{(-1)^3}{3} + b\frac{(-1)^2}{2} + c(-1)) = a(1/3+1/3) = 2a/3 + 2c.$$

    For b) a primitive is $F(x) = -\frac{1}{2}\exp(-2x)$. Hence,

    $$\int_1^2 \exp(-2x)dx = -\frac{1}{2}\exp(-4) + \frac{1}{2}\exp(-2).$$

19. Let $f \in \mathcal{R}[a,b]$ be either positive or negative and let $v \in C([a,b])$. Then, there exists some $\xi \in (a,b)$ such that
$$\int_a^b u(t)v(t)dt = v(\xi)\int_a^b u(t)dt.$$

20. Let $[c,d] \subset [a,b]$. Then $\mathcal{R}[a,b] \subset \mathcal{R}[c,d]$, in the sense that every $f \in \mathcal{R}[a,b]$ satisfies $f$ restricted to $[c,d] \in \mathcal{R}[c,d]$. Moreover,
$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$$