# HOMEWORK 3

## QUESTION 1.

1. $E(r_t) = \mu$, $V(r_t) = \sigma^2$, $\text{cov}(r_t, r_{t\pm h}) = \gamma(h)$.

That is, expectation and variance do not depend on "$t$" and the covariance of any two components of the sequence $\{r_t\}$, depends only on how far apart they are.

2. For any vector $t^T = (t_1 \dots t_n)$, let $F_t$ be the distribution of the vector $(r_{t_1} \dots r_{t_n})$. Strict stationarity requires that

$$F_t = F_{t\pm h}$$

for any $h^T = (h_1, \dots, h_n)$, where $n \in \mathbb{N}$.

3. Yes, when $E(r_t)$, $V(r_t)$ and $\text{Cov}(r_t, r_{t\pm h})$ exist. Since for $t \in \mathbb{Z}$, $F_t = F_{t+h}$, $E(r_t) = \int r_t \, dF_t$ is a constant for every $t$, similarly for $V(r_t)$. Also, the joint distribution of $r_t$ and $r_{t+h}$ does not depend on $t$, only $h$, hence

$$\text{Cov}(r_t, r_{t+h}) = \gamma(h).$$

4. a)

$$\begin{aligned}
r_t &= \phi_0 + \phi\left(\phi_0 + \phi r_{t-4} + \varepsilon_{t-2}\right) + \varepsilon_t \\
&= \phi_0(1+\phi) + \phi^2 r_{t-4} + \phi\varepsilon_{t-2} + \varepsilon_t \\
&= \phi_0(1+\phi) + \phi^2(\phi_0 + \phi r_{t-6} + \varepsilon_{t-4}) + \phi\varepsilon_{t-2} + \varepsilon_t \\
&= \phi_0(1+\phi+\phi^2) + \phi^3 r_{t-6} + \phi^2\varepsilon_{t-4} + \phi\varepsilon_{t-2} + \varepsilon_t
\end{aligned}$$

Repeating the number of lag substitutions to infinity, we have

$$r_t = \phi_0 (1 + \phi + \phi^2 + \cdots) + (\varepsilon_t + \phi\,\varepsilon_{t-2} + \phi^2 \varepsilon_{t-4} + \cdots)$$

where $\lim_{m\to\infty} \phi^{m+1} r_{t-2(m+1)} = 0$ since $|\phi| < 1$

Hence, $$r_t = \frac{\phi_0}{1-\phi} + \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-2j}$$

b) $$V(r_t) = E\left(r_t - \frac{\phi_0}{1-\phi}\right)^2 = E\left(\sum_{j=0}^{\infty} \phi^j \varepsilon_{t-2j}\right)^2$$

$$= \sigma^2 / (1-\phi^2)$$

$$\operatorname{cov}(r_t, r_{t-1}) = E\left(r_t - \frac{\phi_0}{1-\phi}\right)\left(r_{t-1} - \frac{\phi_0}{1-\phi}\right)$$

$$= E\left(\sum_{j=0}^{\infty} \phi^j \varepsilon_{t-2j}\right)\left(\sum_{j=0}^{\infty} \phi^j \varepsilon_{t-1-2j}\right)$$

$$= 0$$

$$\operatorname{cov}(r_t, r_{t-2}) = E\left(\sum_{j=0}^{\infty} \phi^j \varepsilon_{t-2j}\right)\left(\sum_{j=0}^{\infty} \phi^j \varepsilon_{t-2-2j}\right).$$

$$= \phi\sigma^2 + \phi^3\sigma^2 + \phi^5\sigma^2 + \cdots$$

$$= \phi\sigma^2 / 1-\phi^2$$

c) It is different. One difference is that there is no covariance between $r_t$ and $r_{t-1}$. Yes, it is covariance stationary.

d) $\hat{\phi} = \underset{\phi}{\text{argmin}} \sum_{t=1}^{n} (r_t - \phi r_{t-2})^2 = \underset{\phi}{\text{argmin}}\ S_n(\phi)$

Hence, $\frac{d}{d\phi} S_n(\hat{\phi}) = 0$, gives $2 \sum_{t=1}^{n} (r_t - \hat{\phi} r_{t-2})(-r_{t-2}) = 0$

which implies $\hat{\phi} = \frac{\sum_{t=1}^{n} r_t r_{t-2}}{\sum_{t=1}^{n} r_{t-2}^2}$. Now,

$$\hat{\phi} = \frac{\sum_{t=1}^{n} r_{t-2}(\phi r_{t-2} + \varepsilon_t)}{\sum_{t=1}^{n} r_{t-2}^2} = \phi + \frac{\sum_{t=1}^{n} r_{t-2}\varepsilon_t}{\sum_{t=1}^{n} r_{t-2}^2}$$

$E(\hat{\phi} \mid r_{-1}, r_0, \ldots, r_{n-2}) = \phi$, since $E(\varepsilon_t \mid r_{-1}, r_0, \ldots r_{n-2}) = 0$

e) $V(\hat{\phi}) = E((\hat{\phi} - \phi)^2 \mid r_{-1}, r_0, \ldots, r_{n-2})$

$$= E\left(\left[\frac{\sum_{t=1}^{n} r_{t-2}\varepsilon_t}{\sum_{t=1}^{n} r_{t-2}^2}\right]^2 \Bigg| r_{-1}, r_0, \ldots r_{n-2}\right)$$

$$= \sigma^2 \Big/ \sum_{t=1}^{n} r_{t-2}^2$$

QUESTION 2:

1. The dimensions are 3x3. Its elements represent the probability limit of $(\frac{1}{n} R^T R)^{-1}$. That is, the matrix gives the asymptotic variances and covariances of the elements of $\sqrt{n}(\hat{\phi} - \phi)$.

2. By Slutsky Theorem, and given that $\sigma^2$ can be consistently estimated by

$$\hat{\sigma}^2 = \frac{1}{n}(\vec{r} - R\hat{\phi})^T(\vec{r} - R\hat{\phi}),$$

we have that

$$\hat{\sigma}^2 \left(\frac{R^T R}{n}\right)^{-1} \xrightarrow{p} \sigma^2 Q^{-1}$$

3. Let $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $p = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then,

$$P\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \text{ and } H_0: \phi_1 = \phi_2 = 0 \Leftrightarrow H_0: P\phi = p$$

Then, as shown in class. (Lecture 12)

$$t_n = n(P\hat{\phi})^T \left(\hat{\sigma}^2 P \left(\frac{R'R}{n}\right)^{-1} P^T\right)^{-1} P\hat{\phi} \xrightarrow{d} \chi^2_2$$

If desire a $\alpha \in (0,1)$ level of significance, we should obtain the quantile of order $1-\alpha$ for a $\chi^2_2$ and reject $H_0$ if $t_n$ is greater than this quantile.

4// a) No. $R_1$ should be [0 0 1] and on line 21, there should be only 1 degree of freedom, i.e.,

chi2inv (0.95, 1).

b) log returns. The series of prices must be lagged. Hence, sp0 starts at 2 goes to n, and sp1 starts at 1 and goes to n-1.

c) Yes, it is correct. It corresponds to the first lag of the sequence of returns.

QUESTION 3:

See code ar1_apple_homework3.m

QUESTION 4:

See code ar2_apple_homework3_2.m