HOMEWORK 4.

1. Since $E(a_t | a_{t-1}) = 0$, $E(a_t) = 0$ and $V(a_t | a_{t-1}) = 1 + 0.5 a_{t-1}^2$

$V(a_t) = 1 + 0.5\, E(a_{t-1}^2)$.

- If $\{a_t\}$ is covariance stationary, then $V(a_t) = \frac{1}{1-0.5}$

- From the class notes, if $E(a_t | a_{t-1}) = 0$, then $Cov(a_t, a_{t-h}) = \gamma(h) = 0$ for $h = 1, 2, \ldots$

Hence, r_t is an AR(1) process with noise $\{a_t\}$ such that $E(a_t) = 0$, $V(a_t) = 2$ and $\gamma(h) = 0$ for $h \geq 1$. Hence

(a) $E(r_t) = \frac{3}{1-0.7} = 10$ since $|0.7| < 1$.

(b) $V(r_t) = \frac{V(a_t)}{1-(0.7)^2} = 3.92$

(c) $E(r_t - E(r_t))(r_{t-1} - E(r_{t-1})) = 0.7\, V(r_t)$

$E(r_t - E(r_t))(r_{t-k} - E(r_{t-k})) = (0.7)^k\, V(r_t)$

d) $a_t = (\alpha_0 + \alpha_1 a_{t-1}^2)^{1/2} \epsilon_t$

$$E(a_t^2 \mid a_{t-1}, \dots) = \alpha_0 + \alpha_1 a_{t-1}^2$$

Now,

$$a_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + a_t^2 - E(a_t^2 \mid a_{t-1}, \dots)$$
$$= \alpha_0 + \alpha_1 a_{t-1}^2 + v_t$$

where $v_t = a_t^2 - E(a_t^2 \mid a_{t-1} \dots) = (\alpha_0 + \alpha_1 a_{t-1}^2)(\epsilon_t^2 - 1)$

Note that $\{a_t^2\}$ has an AR(1) structure with noise v_t.

$E(v_t \mid a_{t-1}, \dots) = (\alpha_0 + \alpha_1 a_{t-1}^2) E(\epsilon_t^2 - 1) = 0$. Consequently

$E(v_t) = E(E(v_t \mid a_{t-1}, \dots)) = 0$ and $\gamma_{v_t}(h) = 0$ for $h = 1, 2 \dots$ since the conditional expectation $E(v_t \mid a_{t-1} \dots) = 0$.

$$E(v_t^2) = V(v_t) = E(E(v_t^2 \mid a_{t-1}))$$
$$= E\left((\alpha_0 + \alpha_1 a_{t-1}^2)^2 E((\epsilon_t^2 - 1)^2)\right)$$
$$= E\left((\alpha_0 + \alpha_1 a_{t-1}^2)^2\right) E\left((\epsilon_t^2 - 1)^2\right)$$

$$E\left((\alpha_0 + \alpha_1 a_{t-1}^2)^2\right) = E(\alpha_0^2 + 2\alpha_0\alpha_1 a_{t-1}^2 + \alpha_1^2 a_{t-1}^4)$$
$$= \alpha_0^2 + 2\alpha_0\alpha_1 E(a_{t-1}^2) + \alpha_1^2 E(a_{t-1}^4)$$
$$= \alpha_0^2 + 2\alpha_0\alpha_1 \frac{\alpha_0}{1-\alpha_1} + \alpha_1^2 E(a_{t-1}^4)$$
$$= \alpha_0^2 + \frac{2\alpha_1\alpha_0^2}{1-\alpha_1} + \alpha_1^2 E(a_{t-1}^4)$$

Thus, $V(v_t) = \left(\alpha_0^2 + \frac{2\alpha_1\alpha_0^2}{1-\alpha_1} + \alpha_1^2 E(a_{t-1}^4)\right) V(\epsilon_t^2)$

If $V(\epsilon_t^2)$, $E(a_{t-1}^4)$ exist $V(v_t) < \infty$ Hence, a_t^2 is an AR(1) with noise v_t which is uncorrelated with variance of $V(v_t)$. Consequently, from our work with AR(1) processes

$$r_{a_t^2}(h) = \frac{1}{(1-0.5^2)} 0.5^h V(v_t)$$

2.
$$r_1 - 0.7 = \phi(r_0 - 0.7) + a_1$$
$$0.3 = -0.25 + a_1 \Rightarrow a_1 = 0.55$$

$$r_2 - 0.7 = 0.5(1-0.7) + a_2$$
$$= 0.15 + \varepsilon_2 (1 + 0.3 a_1^2)^{1/2}$$

$$\Rightarrow r_2 = 0.85 + 1.0908 \varepsilon_2$$

$$E(r_2 \mid r_0 = 0.2, r_1 = 1) = 0.85$$

$$V(r_2 \mid r_0 = 0.2, r_1 = 1) = 1.0908 \times V(\varepsilon_2) = 1.0908$$

3\. a) For (a) use code arma_garch_apple.m, model 1 on code lines 21 to 24.

(b) use the same code as (a), but model 2 on lines 27 to 30. On line 28, change "dists" to "distg"

(c) Use model 2 on the same code, but change ('ARLags',1) to ('ARLags',2) and ('GARCHLags',1) to ('GARCHLags',2)

b) Just verify if the TStatistic associated with GARCH{1} on the MATLAB output is greater than or equal to 1.96. If so, reject $H_0: \beta_1 = 0$.

c) Same as b), but look at the t-statistic at the ARIMA table from MATLAB. Compare to the 90% quantile of a normally distributed random variable.

4// a) Use code garch_student1.m, model 1, lines 28-36

b) " " " , model 2, lines 38-46

5// a), b), c) answers are directly from class notes (as derived on the board) with N=2. That is,

a) $\hat{w} = \bar{w}\,C_p$, C_p and $\bar{w}$ are defined in your class notes. notes

b) $\hat{w}_f = 1 - \sum_{i=1}^{2} \hat{w}_i$

c) $w_T = \bar{w} / \sum_{i=1}^{2} \bar{w}_i$

d) $\hat{\mu}_1 = \frac{1}{n} \sum_{t=1}^{n} R_{t,A}$, $\hat{\mu}_2 = \frac{1}{n} \sum_{t=1}^{n} R_{t,B}$.

$$Cov(R_{t,A}, R_{t,B}) = \frac{1}{n} \sum_{t=1}^{n} (R_{tA} - \hat{\mu}_1)(R_{tB} - \hat{\mu}_2)$$

$$V(R_{t,A}) = \frac{1}{n} \sum_{t=1}^{n} (R_{tA} - \hat{\mu}_1)^2$$

$$V(R_{t,B}) = \frac{1}{n} \sum_{t=1}^{n} (R_{tB} - \hat{\mu}_2)^2$$

e) Yes. Since $\{R_{t,A}\}$ is assumed to be IID.

$$E(\hat{\mu}_1) = \frac{1}{n} n \cdot \mu_1 = \mu_1$$